ASTR 702
Stellar Models (Chapter 5)

This chapter of the book deals with some analytical solutions to the equations of stellar structure that were developed before the advent of powerful computers. As such, these solutions feel a little outdated, but they do lead to important equations for the Eddington Limit and the Chandrasekhar Mass.

In equilibrium, our equations are:

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \tag{1}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{2}$$

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T^3}\frac{F}{4\pi r^2} \tag{3}$$

$$\frac{dF}{dr} = 4\pi r^2 \rho q \tag{4}$$

or in terms of mass

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \tag{5}$$

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho} \tag{6}$$

$$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\bar{\kappa}}{T^3}\frac{F}{4\pi r^2}^{\;2} \tag{7}$$

$$\frac{dF}{dm} = q \tag{8}$$

We also have equations for pressure, opacity, and heat generation:

$$P = \frac{R}{\mu_I}\rho T + P_e + 1/3aT^4 \tag{9}$$

$$\kappa = \kappa_0 \rho^a T^b \tag{10}$$

$$q = q_0 \rho^m T^n \tag{11}$$

Finally, we have the boundary conditions of $m = 0$ at $r = 0$, $P = 0$ at $r = R$.

Polytropic Models

We know that pressure depends on temperature, and that relationship is what links Equations 1-2 to Equations 3-4. If we assume, however, that the pressure does not depend on the temperature, but instead only on the density, we have "polytropic" solutions. They are also called "homology relations." These solutions will make use of the polytropic equation we had earlier:

$$P = K\rho^\gamma\,, \tag{12}$$

with $\gamma = 1 + 1/n$.

We had before two polytropes, one for degenerate electrons and one for degenerate relativistic electrons:

$$P_{\rm e,deg} = K_1^{\text{‘}} \left(\frac{\rho}{\mu_e}\right)^{5/3}, \tag{13}$$

with $K_1^{\text{‘}} = 1.00 \times 10^7\,{\rm m}^4\,{\rm kg}^{2/3}\,{\rm s}^{-2}$ and

$$P_{\rm e,deg} = K_2^{\text{‘}} \left(\frac{\rho}{\mu_e}\right)^{4/3}, \tag{14}$$

with $K_2^{\text{‘}} = 1.24 \times 10^{10}\,{\rm m}^3\,{\rm kg}^{-1/3}\,{\rm s}^{-1}$.

If we multiply the equation of hydrostatic equilibrium by r^2/ρ and differentiate with respect to r, we find

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr} \tag{15}$$

We can then substitute in mass conservation to get

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho\,. \tag{16}$$

and finally if we substitute in the polytropic equation we get

$$\frac{(n+1)K}{4\pi G n}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{(n-1)/n}}\frac{d\rho}{dr}\right) = -\rho \tag{17}$$

The solution for $\rho(r)$ is called a polytrope. We need two boundary conditions: $\rho = 0$ at $r = R$ and $d\rho/dr = 0$ at $r = 0$. Therefore, our polytrope relation tells us that ρ is defined by just three parameters: $K, n,$ and R.

In what follows next, we will define some auxiliary variables to reduce this expression. First, we will define θ:

$$\rho = \rho_c \theta^n\,, \tag{18}$$

and $0 \leq \theta \leq 1$. θ therefore tells us the difference between ρ and ρ_c. Substituting in, we find

$$\left[\frac{(n+1)K}{4\pi G\rho_c^{(n-1)/n}}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n \tag{19}$$

We know that the RHS has no units. The term in brackets must have dimensions of length squared, and be a constant. We can therefore write

$$\alpha^2 = \left[\frac{(n+1)K}{4\pi G\rho_c^{(n-1)/n}}\right], \tag{20}$$

so that α has units of length. We can therefore define

$$\xi = r/\alpha\,. \tag{21}$$

The meaning of these auxiliary variables is a bit hard to follow, so let's review. θ tells us how ρ differs from ρ_c and varies from 0 to 1. α is a constant with dimensions of length, and ξ tells us how r and α are related. Mashing this all together, we get the famous Lane-Emden equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n\,. \tag{22}$$

The boundary equations on this say that $\theta = 1$ at $\xi = 0$ ($r = 0$ and $\rho = \rho_c$) and $d\theta/d\xi = 0$ at $\xi = 0$ ($d\rho/dr = 0$ at $r = 0$).

Whew! We can integrate the Lane-Emden equation for a given n. These solutions show that if n is large, the density peaks more strongly in the center.

We can use these relations to rewrite the mass of a polytropic star:

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi\alpha^3\rho_c \int_0^{\xi_1} \xi^2\theta_n d\xi\,, \tag{23}$$

where $\xi_1 = R/\alpha$. Using the Lane-Emden equation, we can write

$$M = 4\pi\alpha^3\rho_c \int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) d\xi = 4\pi\alpha^3\rho_c\xi_1^2\left(\frac{d\theta}{d\xi}\right)|_{\xi_1} \tag{24}$$

But that's not all! Let's make some more variables! As we did for θ, let's define how far the central density is from the average density. This parameter, D_n, varies with n.

$$\rho_c = D_n\bar{\rho} = D_n\frac{M}{4\pi/3R^3}\,, \tag{25}$$

and therefore

$$D_n = -\left[\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)|_{\xi_1}\right]^{-1} \tag{26}$$

We can then eliminate ρ_c and substitute α to get a relationship between mass and radius that only depends on the constant $M_n = -\xi^2(d\theta/d\xi)$ and $R_n = \xi_1$:

$$\left(\frac{GM}{M_n}\right)^{n-1}\left(\frac{R}{R_n}\right)^{3-n} = \frac{[(n+1)K]}{4\pi G}\,. \tag{27}$$

Finally, we can get a relationship between the central pressure and our terms:

$$P_c = \frac{(4\pi G)^{1/n}}{n+1}\left(\frac{GM}{M_n}\right)^{(n-1)/n}\left(\frac{R}{R_n}\right)^{(3-n)/n}\rho_c^{(n+1)/n} \tag{28}$$

and if we collect all coefficients that depend on n into yet anotehr variable B_n,

$$P_c = (4\pi)^{1/3}B_n G M^{2/3}\rho_c^{4/3}\,. \tag{29}$$

OK, that's quite enough! The amazing thing about these relations is that they all depend on the polytropic index n. If we can determine that, we can solve for relations of all the other stellar properties.

The Chandrasekhar Mass

One application of the polytropic models that we can assume best represents reality is for degenerate stars. We said before that for these stars there is no dependence of the pressure on the temperature.

If we take

$$\left(\frac{GM}{M_n}\right)^{n-1}\left(\frac{R}{R_n}\right)^{3-n} = \frac{[(n+1)K]}{4\pi G}\,, \tag{30}$$

and $n = 1.5$ ($\gamma = 5/3$) for degenerate material, we find $R \propto M^{-1/3}$. The density $\rho \propto MR^{-3} \propto M^2$. Therefore, for degenerate stars, as the mass goes up, the radius goes down and the density goes up. In other words, higher mass white dwarfs are smaller.

If the density is high enough, the electrons are relativistic, and $n = 3$ ($\gamma = 4/3$). This has a special form, since $\left(\frac{R}{R_n}\right)^{3-n} = 1$ and therefore the mass is independent of radius:

$$M = 4\pi M_3\left(\frac{K}{\pi G}\right)^{3/2}\,. \tag{31}$$

Therefore, the only solution for M depends on the value of K. If we take $K = K_2'$ from before, we find

$$M_{\rm Ch} = \frac{M_3\sqrt{1.5}}{4\pi}\left(\frac{hc}{Gm_H^{4/3}}\right)^{3/2}\mu_e^{-2} = 5.83\mu_e^{-2}\,M_\odot \tag{32}$$

Reasonable values of μ_e range from $\mu_e = 2$ (He, C, O) to $\mu_e = 2.15$ (Fe), which gives us 1.46 $M_\odot$ to 1.26 $M_\odot$. This is the maximum mass of a white dwarf.

The Eddington Luminosity

I have stated a few times now, without evidence, that stars with masses $M \gtrsim 100\ M_\odot$ will blow themselves apart. If we substitute in the expression for radiation pressure, $P_{\rm rad} = 1/3aT^4$, and then substitute hydrostatic equilibrium into the temperature gradient equation, we find

$$\frac{dP_{\rm rad}}{dP} = \frac{\kappa F}{4\pi cGm} \tag{33}$$

Since $P = P_{\rm gas} + P_{\rm rad}$, $dP_{\rm rad} < dP$ and

$$\kappa F < 4\pi cGm\,, \tag{34}$$

Near the center of the star, $F(0) = 0$ so $F/m \to q_c$ as $m \to 0$ where $q_c = q(m = 0)$. Therefore, radiative energy transfer can only accommodate:

$$q_c < \frac{4\pi cG}{\kappa}\,. \tag{35}$$

If we then substitute in for q,

$$L < \frac{4\pi cGM}{\kappa} \tag{36}$$

and the so-called maximum luminosity before hydrostatic equilibrium is violated

$$L_{\mathrm{Edd}} = \frac{4\pi c G M}{\kappa} = 3.2 \times 10^4 \left(\frac{M}{M_\odot}\right) \left(\frac{\kappa_{\mathrm{es}}}{\kappa}\right) L_\odot \tag{37}$$

When combined with the mass-luminosity relation, there is an upper limit to the mass of a main sequence star.