Stellar
Stellar Models

We can solve the equations of stellar structure analytically, after some assumptions. This method was developed before the advent of powerful computers. As such, these solutions feel a little outdated, but they do lead to important equations for the Eddington Limit and the Chandrasekhar Mass, so let's divein.

In equilibrium, our equations are:

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \tag{1}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{2}$$

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T^3}\frac{F}{4\pi r^2} \tag{3}$$

$$\frac{dF}{dr} = 4\pi r^2 \rho q \tag{4}$$

or in terms of mass

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \tag{5}$$

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho} \tag{6}$$

$$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\bar{\kappa}}{T^3}\frac{F}{4\pi r^2}^2 \tag{7}$$

$$\frac{dF}{dm} = q \tag{8}$$

We also have equations for the equation of state, opacity, and heat generation:

$$P = \frac{k}{\mu_I m_H}\rho T + P_e + 1/3aT^4 \tag{9}$$

$$\kappa = \kappa_0 \rho^a T^b \tag{10}$$

$$q = q_0 \rho^m T\,, \tag{11}$$

where P_e could be from thermal, degenerate, or relativistic degenerate. Finally, we have the boundary conditions of $m = 0$ at $r = 0$, $P = 0$ at $r = R$. Regardless of our method (analytical or computational), these are the equations that we need to use.

Polytropic Models

Before computers, astronomers had to derive expressions analytically, and the polytropic models are a relic from that time. They are still useful, however, for building an understanding of the physics.

We know that pressure depends on temperature, and that relationship is what links Equations 1-2 to Equations 3-4. This dependence, however, makes deriving analytical expressions appropriate for stellar interiors difficult. If we assume that the pressure does not depend on the temperature but instead only on the density (as we have seen for degenerate gas), OR that the temperature and pressure are related by $T \propto \rho^{1/n}$ for some index n, we have "polytropic" solutions. These solutions will make use of the polytropic equation we had earlier:

$$P = K\rho^{1+1/n} = K\rho^{\gamma}\,, \tag{12}$$

with $\gamma = 1 + 1/n$.

We had before two polytropes, one for degenerate electrons and one for degenerate relativistic electrons:

$$P_{\mathrm{e,deg}} = K_1^{'}\left(\frac{\rho}{\mu_e}\right)^{5/3}\,, \tag{13}$$

with $K_1^{'} = 1.00 \times 10^{7}\,\mathrm{m}^4\,\mathrm{kg}^{2/3}\,\mathrm{s}^{-2}$ and

$$P_{\mathrm{e,deg}} = K_2^{'}\left(\frac{\rho}{\mu_e}\right)^{4/3}\,, \tag{14}$$

with $K_2^{'} = 1.24 \times 10^{10}\,\mathrm{m}^3\,\mathrm{kg}^{-1/3}\,\mathrm{s}^{-1}$.

We can derive the structure of a polytrope if we multiply the equation of hydrostatic equilibrium by r^2/ρ and differentiate with respect to r. We find

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr} \tag{15}$$

We can then substitute in mass conservation to get

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho\,. \tag{16}$$

We know from the polytropic equation 12

$$\frac{dP}{dr} = K(1+1/n)\rho^{1/n}\frac{d\rho}{dr} = \frac{K(n+1)}{n}\rho^{1/n}\frac{d\rho}{dr} \tag{17}$$

and finally if we substitute in we get

$$\frac{1}{4\pi G}\frac{(n+1)K}{n}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{(n-1)/n}}\frac{d\rho}{dr}\right)=-\rho \tag{18}$$

The solution for $\rho(r)$ is called a polytrope. We need two boundary conditions: $\rho = 0$ at $r = R$ and $d\rho/dr = 0$ at $r = 0$. Therefore, our polytrope relation tells us that ρ is defined by just three parameters: $K, n,$ and R.

OK, so far, so good, but the choice of variables in the following will take some getting used to. We will define some auxiliary variables to reduce this expression. First, we will define θ:

$$\rho = \rho_c \theta^n\,, \tag{19}$$

and put bounds on θ: $0 \le \theta \le 1$. θ therefore tells us the difference between ρ and ρ_c. Substituting in, after some manipulation we find

$$\left[\frac{(n+1)K}{4\pi G\rho_c^{(n-1)/n}}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right)=-\theta^n \tag{20}$$

We know that the RHS has no units. The term in brackets must have dimensions of length squared, and be a constant. We can therefore write

$$\alpha^2 = \left[\frac{(n+1)K}{4\pi G\rho_c^{(n-1)/n}}\right], \tag{21}$$

so that α has units of length. We can define

$$\xi = r/\alpha\,. \tag{22}$$

The meaning of these auxiliary variables is a bit hard to follow, so let's review. θ tells us how ρ differs from ρ_c and varies from 0 to 1. α is a constant with dimensions of length, and ξ tells us how r and α are related. Mashing this all together, we get the famous Lane-Emden equation:

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right)=-\theta^n}\,. \tag{23}$$

The boundary equations on this say that $\theta = 1$ at $\xi = 0$ ($r = 0$ and $\rho = \rho_c$) and $d\theta/d\xi = 0$ at $\xi = 0$ ($d\rho/dr = 0$ at $r = 0$). It can only be solved analytically for $n = 0, 1,$ and 5.

Whew! We can integrate the Lane-Emden equation for a given n. These solutions show that if n is large, the density peaks more strongly in the center.

We can use these relations to rewrite the mass of a polytropic star:

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi\alpha^3\rho_c \int_0^{\xi_1} \xi^2\theta_n d\xi\,, \tag{24}$$

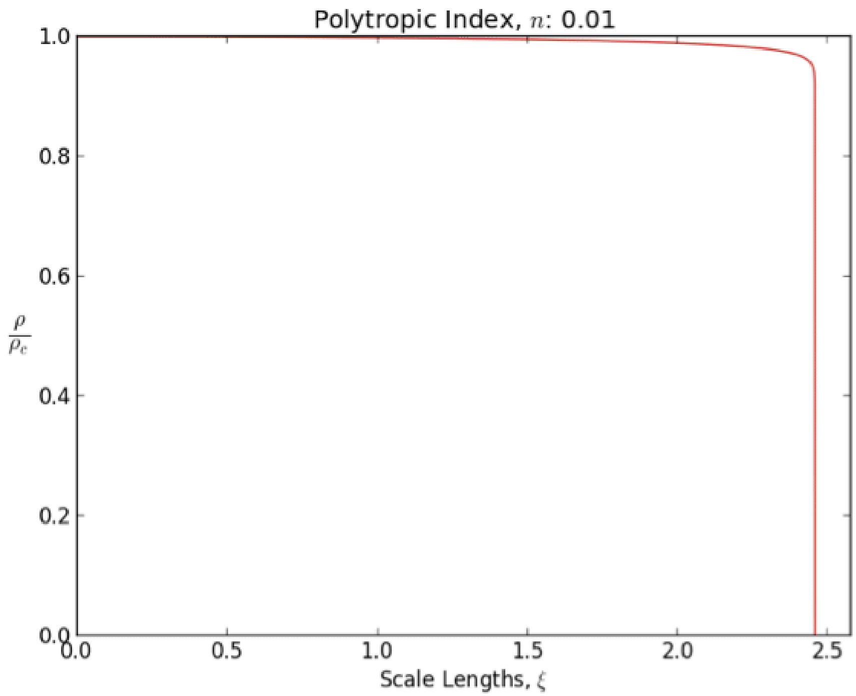

Figure 1: Dependence of density on polytropic index, from Wikipedia

where $\xi_1 = R/\alpha$. Using the Lane-Emden equation, we can write

$$M = 4\pi\alpha^3\rho_c \int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi = 4\pi\alpha^3\rho_c \xi_1^2 \left(\frac{d\theta}{d\xi}\right)\bigg|_{\xi_1} \tag{25}$$

But that's not all! Let's make some more variables! As we did for θ, let's define how far the central density is from the average density. This parameter, D_n, varies with n.

$$\rho_c = D_n \bar{\rho} = D_n \frac{M}{4\pi/3R^3}, \tag{26}$$

and therefore

$$D_n = -\left[\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)\bigg|_{\xi_1}\right]^{-1} \tag{27}$$

We can then eliminate ρ_c and substitute α to get a relationship between mass and radius that only depends on the constant $M_n = -\xi^2(d\theta/d\xi)$ and $R_n = \xi_1$:

$$\left(\frac{GM}{M_n}\right)^{n-1}\left(\frac{R}{R_n}\right)^{3-n} = \frac{[(n+1)K]}{4\pi G}. \tag{28}$$

Finally, we can get a relationship between the central pressure and our terms:

$$P_c = \frac{(4\pi G)^{1/n}}{n+1}\left(\frac{GM}{M_n}\right)^{(n-1)/n}\left(\frac{R}{R_n}\right)^{(3-n)/n}\rho_c^{(n+1)/n} \tag{29}$$

n	D_n	M_n	R_n	B_n
1.0	3.290	3.14	3.14	0.233
1.5	5.991	2.71	3.65	0.206
2.0	11.40	2.41	4.35	0.185
2.5	23.41	2.19	5.36	0.170
3.0	54.18	2.02	6.90	0.157
3.5	152.9	1.89	9.54	0.145

Figure 2: Values of the Lane-Emden variables, from your book.

and if we collect all coefficients that depend on n into yet another variable B_n,

$$P_c = (4\pi)^{1/3} B_n G M^{2/3} \rho_c^{4/3} \,. \tag{30}$$

OK, that's quite enough! The amazing thing about these relations is that they all depend on the polytropic index n. If we can determine that, we can solve for relations of all the other stellar properties.

The Chandrasekhar Mass

One application of the polytropic models that we can assume best represents reality is for degenerate stars. We said before that for these stars there is no dependence of the pressure on the temperature, so our polytropic assumption is valid.

If we take

$$\left(\frac{GM}{M_n}\right)^{n-1} \left(\frac{R}{R_n}\right)^{3-n} = \frac{[(n+1)K]}{4\pi G} \,, \tag{31}$$

and $n = 1.5$ ($\gamma = 5/3$) for degenerate material, we find $R \propto M^{-1/3}$. The density $\rho \propto MR^{-3} \propto M^2$. Therefore, for degenerate stars, as the mass goes up, the radius goes down and the density goes up. In other words, higher mass white dwarfs are smaller.

If the density is high enough, the electrons are relativistic, and $n = 3$ ($\gamma = 4/3$). This has a special form, since $\left(\frac{R}{R_n}\right)^{3-n} = 1$ and therefore the mass is independent of radius:

$$M = 4\pi M_3 \left(\frac{K}{\pi G}\right)^{3/2} \,. \tag{32}$$

Therefore, the only solution for M depends on the value of K. If we take $K = K_2'$

from before, we find

$$M_{\rm Ch} = \frac{M_3\sqrt{1.5}}{4\pi}\left(\frac{hc}{Gm_H^{4/3}}\right)^{3/2}\mu_e^{-2} = 5.83\mu_e^{-2}\,M_\odot \tag{33}$$

Reasonable values of μ_e range from $\mu_e = 2$ (He, C, O) to $\mu_e = 2.15$ (Fe), which gives us 1.46 $M_\odot$ to 1.26 $M_\odot$. This is the maximum mass of a white dwarf (OK, so rotation can alter things a little more...).

The Eddington Luminosity

I have stated a few times now, without evidence, that stars with masses $M \gtrsim 100\ M_\odot$ will blow themselves apart. If we substitute in the expression for radiation pressure, $P_{\rm rad} = 1/3aT^4$, and then substitute hydrostatic equilibrium into the temperature gradient equation, we find

$$\frac{dP_{\rm rad}}{dP} = \frac{\kappa F}{4\pi cGm} \tag{34}$$

Since $P = P_{\rm gas} + P_{\rm rad}$, $dP_{\rm rad} < dP$ and

$$\kappa F < 4\pi cGm\,, \tag{35}$$

Near the center of the star, $F(0) = 0$ so $F/m \to q_c$ as $m \to 0$ where $q_c = q(m = 0)$. Therefore, radiative energy transfer can only accommodate:

$$q_c < \frac{4\pi cG}{\kappa}\,. \tag{36}$$

If we then substitute in for q,

$$L < \frac{4\pi cGM}{\kappa} \tag{37}$$

and the so-called maximum luminosity before hydrostatic equilibrium is violated

$$L_{\rm Edd} = \frac{4\pi cGM}{\kappa} = 3.2\times10^4\left(\frac{M}{M_\odot}\right)\left(\frac{\kappa_{\rm es}}{\kappa}\right)L_\odot \tag{38}$$

When combined with the mass-luminosity relation, this gives an upper limit to the mass of a main sequence star.