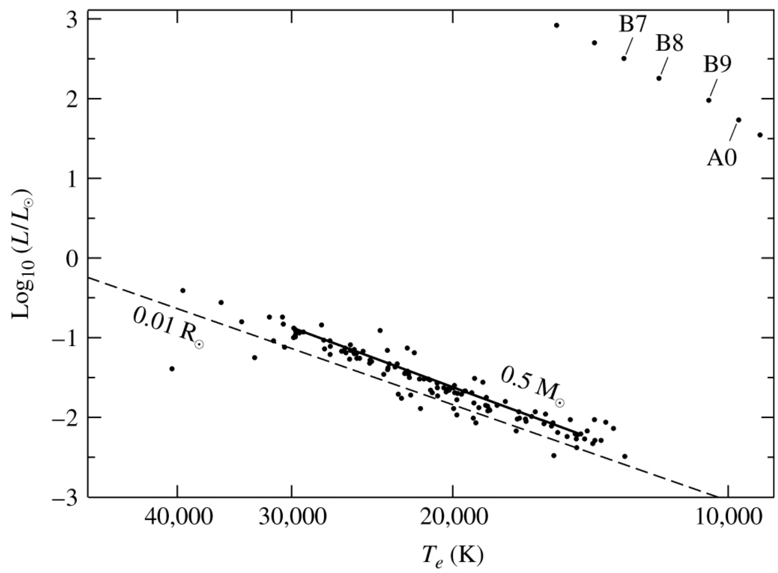

Figure 1: A portion of the H-R diagram showing white dwarfs.

White Dwarfs
C+O Chapter 16

We know what is left behind at the end of stellar evolution. For low-mass stars ($< 8\ M_\odot$), the core is left behind when the stars go planetary nebula. These cores are then white dwarfs: very small, very hot, very low luminosity objects. High-mass stars ($> 8\ M_\odot$) may also leave behind their cores, but the higher pressure results in neutron stars rather than white dwarfs. The most massive stars can become black holes. In this section we will discuss white dwarfs only, with the other objects discussed in the coming weeks.

Exercise

Let's compute the basic properties of white dwarfs! Take a 20,000 K White Dwarf on the 0.5 $M_\odot$ line and compute its
- Radius (from the luminosity and temperature. Earth's radius is $\sim$6000 km)
- Density assuming its mass is 0.5 $M_\odot$ (density of the Earth is $\sim 5\,\mathrm{g\,cm^{-3}}$ and water is $1\,\mathrm{g\,cm^{-3}}$)
- Central pressure (from hydrostatic equilibrium assuming constant density)

Notice that the 0.5 $M_\odot$ line is sloping toward the 0.01 $R_\odot$ line. What does that imply?

The Fermi Energy

The Fermi energy is the energy of the highest occupied state. Fermions will preferentially fill the lowest energy states, but this filling is governed by the Pauli exclusion principle.

The Pauli exclusion principal says that no two *fermions* (spin 1/2 particles such as electrons, protons, and neutrons) can exist in the same quantum state. But what is a quantum state? It's a set of quantum numbers (n, ℓ, m_ℓ, s) and also a location.

For atoms, the location is the atom itself. We cannot have two $s = +1/2$ electrons in $n = 1$, $\ell = 0$, but it can have one with $s = +1/2$ and one with $s = -1/2$. Thus, the $n = 1$ shell has a maximum of two electrons. For $n = 1$, $\ell = 0$ or 1 and $m_\ell = 0$ or $-1, 0, 1$. The $n = 2$ shell thus has 8 electrons ($\ell = 0$ adds 2 and $\ell = 1$ adds 6). You already probably knew about the Pauli exclusion principle from this application.

For a plamsa, it's a bit trickier, but the criterion that no two fermions can share the same quantum state sets the size scale of a deBroglie wavelength. $\lambda = h/p$.

What is the maximum energy per particle at a given temperature? For particles in a box, the deBroglie wavelength in each dimension is $\lambda_x = 2L/N_x$, where the box has length L and N_x is an integer quantum number (that doesn't include spin). y and z are the same. We can rewrite in terms of momemtum since $p = h/\lambda$ to get $p_x = hN_x/(2L)$ and the same for y and z.

Since momentum is $p^2 = p_x^2 + p_y^2 + p_z^2 = h^2(N_x^2 + N_y^2 + N_z^2)/(4L^2)$ and the energy of a particle is $\epsilon = p^2/(2m)$, we have

$$\epsilon = \frac{p^2}{2m} = \frac{h^2}{8mL^2}(N_x^2 + N_y^2 + N_z^2) = \frac{h^2N^2}{8mL^2} \tag{1}$$

Each quantum state is defined by N_x, N_y, N_z and the spin.

The total number of electrons N_e defined by positive integers N_x, N_y, N_z is

$$2\left(\frac{1}{8}\right)\left(\frac{4}{3}\pi N^3\right), \tag{2}$$

where the factor of 2 comes from the fact that quanum states N do not include the spin and there are two possible spin states. Thus

$$N = \left(\frac{3N_e}{\pi}\right)^{1/3} \tag{3}$$

and finally we arrive at the Fermi energy:

$$\epsilon_F = \frac{\hbar^2}{2m}(3\pi^2 n)^{2/3}, \tag{4}$$

where $n \equiv N_e/L^3$ is the number density. We switched to electrons in this derivation but of course the expression applies to all fermions.

For fully degenerate gas at $T = 0$ K, all the fermion energies will be the Fermi energy. For full ionization,

$$n_e = \left(\frac{\#\text{electrons}}{\text{nucleon}}\right)\left(\frac{\#\text{nucleons}}{\text{volume}}\right) = \left(\frac{Z}{A}\right)\frac{\rho}{m_H} \tag{5}$$

Plugging back in to the Fermi energy, we find

$$\epsilon_F = \frac{\hbar^2}{2m}\left[3\pi^2\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{2/3} \tag{6}$$

We can also express $(Z/A)\rho/m_H$ as N/V where N is now the number of particles per volume V:

$$\epsilon_F = \frac{\hbar^2}{2m}\left[3\pi^2\frac{N}{V}\right]^{2/3} \tag{7}$$

We see from this expression that the Fermi energy is inversely related to the mass. Thus for fully degenerate gas, the Fermi energy is $\sim$2000 times higher for protons and neutrons than it is for electrons. The upshot is that electrons will become degenerate before protons or neutrons.

The N'th particle has an energy of

$$E_{N'} = E_0 + \epsilon_\mathrm{F}\big|_{N'} = E_0 + \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N'}{V}\right)^{2/3} \tag{8}$$

The total energy of a Fermi sphere of N fermions is given by:

$$E_\mathrm{T} = NE_0 + \int_0^N \epsilon_\mathrm{F}\big|_{N'}\, dN' = \left(\frac{3}{5}\epsilon_\mathrm{F} + E_0\right)N \tag{9}$$

Therefore, the average energy per particle is given by:

$$E_\mathrm{av} = E_0 + \frac{3}{5}\epsilon_\mathrm{F} \tag{10}$$

From the first law of thermodynamics, this internal energy can be expressed as a pressure

$$P = -\frac{\partial E_\mathrm{T}}{\partial V} = \frac{2}{5}\frac{N}{V}\epsilon_\mathrm{F} = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\left(\frac{N}{V}\right)^{5/3}. \tag{11}$$

This pressure is known as the "degeneracy pressure" (often $n = N/V$ is substituted) or

$$P = \frac{(3\pi^2)^{2/3}\hbar^2}{5m}\left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{5/3}. \tag{12}$$

White Dwarfs

A white dwarf is very dense: its mass is comparable to that of the Sun, while its volume is comparable to that of Earth. A white dwarf's faint luminosity comes from the emission of stored thermal energy; no fusion takes place in a white dwarf and there is no further contraction to release gravitational potential.

The nearest known white dwarf is Sirius B, at 8.6 light years, the smaller component of the Sirius binary star. Because white dwarfs are the end point of all low-mass stars, which are very numerous (over 97% of the other stars in the Milky Way), there should be lots of white dwarfs in the Universe. Because they are faint, however, they are difficult to find.

Usually, white dwarfs are composed of carbon and oxygen and the progenitor mass is less than 8 $M_{\odot}$. Some flavors of white dwarf do have different compositions though.

If the mass of the progenitor is between 8 and 10.5 solar masses, the core temperature will be sufficient to fuse carbon but not neon, in which case an oxygen/neon/magnesium white dwarf may form. Although helium in most white dwarfs could be fused, this isn't always true for low mass stars. Stars of very low mass may accrete He from a binary companion, and so may have He in their outer layers.

The material in a white dwarf no longer undergoes fusion reactions, so the star has no source of energy. As a result, it cannot support itself by the heat generated by fusion against gravitational collapse, but is supported only by electron degeneracy pressure, causing it to be extremely dense. A white dwarf is very hot when it forms, but because it has no source of energy, it will gradually cool as it radiates its energy. This means that its radiation, which initially has a high color temperature, will lessen and redden with time. Over a very long time, a white dwarf will cool and its material will begin to crystallize, starting with the core. The star's low temperature means it will no longer emit significant heat or light, and it will become a cold black dwarf. Because the length of time it takes for a white dwarf to reach this state is calculated to be longer than the current age of the universe, it is thought that no black dwarfs yet exist. The oldest white dwarfs still radiate at temperatures of a few thousand kelvins. hite dwarfs have an extremely small surface area to radiate this heat from, so they cool gradually, remaining hot for a long time.

Types of White Dwarfs

Stamp collecting!
White Dwarfs can be classified based on their spectra. DA white dwarfs have hydrogen absorption lines in their spectra. These lines are extremely pressure broadened due to the high pressures in the white dwarf surfaces. White dwarfs are mostly carbon and oxygen, but some do have trace amounts of hydrogen. 2/3 of all WDs.
DB white dwarfs have helium absorption lines ,but lack hydrogen. 8% of all WDs.

DC white dwarfs have no lines, they are black bodies. 14% of WDs.

Conditions for White Dwarfs

How big are white dwarfs? From Stefan-Boltzmann,$L = 4\pi R^2 \sigma T^4$. How can we estimate the temperature? Using the colors! Let's assume $T = 10^4$ K and $L = 0.01\ L_\odot$

How can we get the pressure? Using the above degeneracy pressure (which we assume is applicable), we get $\sim 2 \times 10^{22}\,\mathrm{N\,m^{-2}}$. We can also use the expression derived in-class for constant density:

$$P_c \approx 2/3\pi G \rho^2 R^2 \tag{13}$$

For the mass of the Sun and radius of the Earth, this works out to $P_c \approx 4 \times 10^{22}\,\mathrm{N\,m^{-2}}$, which is $\sim 1.5 \times 10^6$ times that of the Sun.

How about temperatures? We can make similar arguments for non-convective stars:

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2} \tag{14}$$

Your book notes that it is not radiation that carries energy to the surface, but rather electron conduction. Oh well, close enough. The temperatures go from the surface temp to the central temperature. The radius goes from the radius of the white dwarf to zero, so

$$\frac{T_{wd} - T_c}{R - 0} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T_c^3}\frac{L_r}{4\pi R^2} \tag{15}$$

We can get the surface temperature from observations, and R from the above calculations. If, however, we assume the surfact temperature is 0 K and that $\kappa = 0.02\,\mathrm{m^2\ kg^{-1}}$, and we get $T_c \approx 10^7 - 10^8$ K. This temperature is plenty high for hydrogen fusion. Since white dwarfs do not have fusion, we know that they must be largely devoid of hydrogen. What little hydrogen they have is on the surface; the more massive elements are drawn toward white dwarf cores.

White Dwarf Masses

We know that white dwarfs have masses of approximately that of the Sun. Obviously the white dwarf that is produced when the Sun reaches the end of its evolution will not be 1 $M_\odot$ though. The most common mass is $\sim 0.5\ M_\odot$.

Electron degeneracy pressure is

$$P = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}\left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{5/3} . \tag{16}$$

As we derived before, the central pressure is roughly

$$P_c \approx \frac{2}{3}\pi G\rho^2 R^2\,. \tag{17}$$

Remember, this equation assumed a constant density throughout the star or white dwarf.

Let's equate them! The first pressure expression tells us the dependence of electron degeneracy pressure on the density. The second one tells us how the central pressure depends on density and radius. This will allow us to find a relationship between the mass and radius.

$$\frac{2}{3}\pi G\rho^2 R^2 = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}\left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{5/3}\,, \tag{18}$$

which for constant density reduces to

$$R \approx \frac{(18\pi)^{2/3}}{10}\frac{\hbar^2}{G m_e M^{1/3}}\left[\left(\frac{Z}{A}\right)\frac{1}{m_H}\right]^{5/3} \tag{19}$$

For a 1 $M_\odot$ white dwarf, $R \approx 2.9 \times 10^6\,\mathrm{m}$, which is too small by a factor of two, so our asssumptions are not great. But, the weird (and correct) thing is that

$$MV = \text{constant}\,, \tag{20}$$

where V is the volume. This is what we saw in the H-R diagram. The mass and volume of a WD are inversely correlated: as the mass increases the radius decreases. This is a result of the odd way that electron degeneracy works. The electrons must be more closely confined to generate the larger degeneracy pressure required to support a more massive star. Of course, this relationship doesn't hold at the extremes, because as we add more and more mass the radius would tend toward zero.

The Chandrasekhar Limit

One way to resolve this issue is to derive the maximum mass a white dwarf can have. If we set the pressure estimate equal to the degeneracy pressure and replace the density with $\rho = M/(4/3\pi R^3)$, we can simplify to get

$$M_{\max} \approx \frac{3\sqrt{2\pi}}{8}\left(\frac{\hbar c}{G}\right)^{3/2}\left[\left(\frac{Z}{A}\right)\frac{1}{m_H}\right]^2 = 0.44\ M_\odot\,, \tag{21}$$

if $Z/A = 0.5$. A precise derivation finds $M = 1.44\ M_\odot$ for $Z/A = 0.5$, which is known as the "Chandrasekhar Limit," after the brilliant Indian astrophysicist.

This is the criterion where hydrostatic equilibrium holds, where degeneracy pressure can just balance gravitational pressure. A mass larger than this an gravitational pressure will "win", collapsing the white dwarf. Therefore, the largest mass white dwarfs should be 1.44 $M_\odot$, or thereabouts after accounting for spin and metallicity.

Note that there is no temperature dependence on all of this! Electron degeneracy pressure doesn't depend on the temperature.

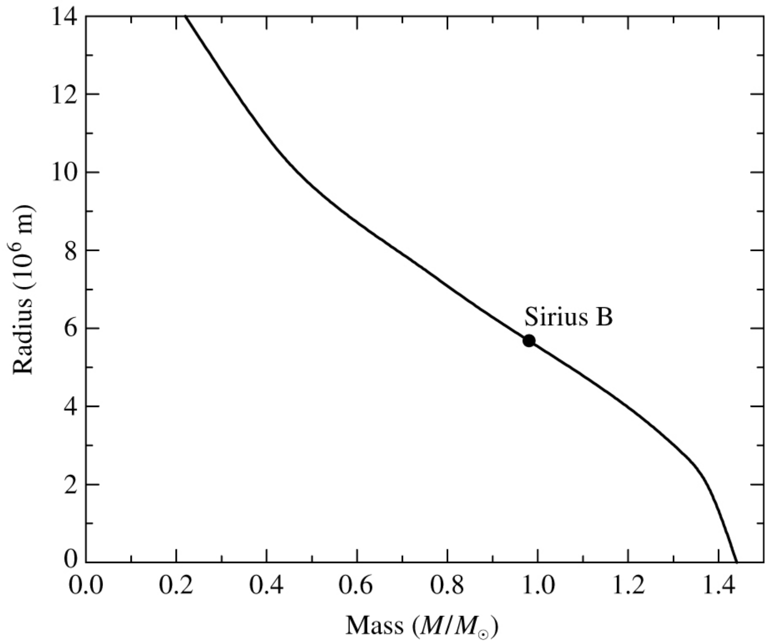

Figure 2: The white-dwarf mass-volume relationship.

White Dwarf Cooling

Energy in white dwarfs does not escape most efficiently from photons. In fact, it is electron conduction that provides the dominant energy transportation method. In a WD, electrons can travel large distances before interacting with another nucleus. As a result, WDs are basically isothermal. The only place that is not isothermal is the outer shell of material.

Your book derives an expression for the WD luminosity, but it's kind of a tough derivation and not terribly informative. The result is interesting though:

$$L = CT_c^{7/2}\,, \tag{22}$$

where C is a constant that is

$$C = 6.65 \times 10^{-3} \left(\frac{M}{M_\odot}\right) \frac{\mu}{Z(1+X)} \tag{23}$$

So how does a WD cool? Well, the WD's energy is thermal, and each nucleus has $3/2kT$ of energy. Therefore, the thermal energy of a WD is

$$U = \frac{M}{Am_H}\frac{3}{2}kT_c\,, \tag{24}$$

where Am_H is the mass of one nucleus. The characteristic timescale (not the cooling time) is

$$\tau = \frac{U}{L} = \frac{3}{2}\frac{Mk}{Am_h C T_c^{5/2}} \tag{25}$$

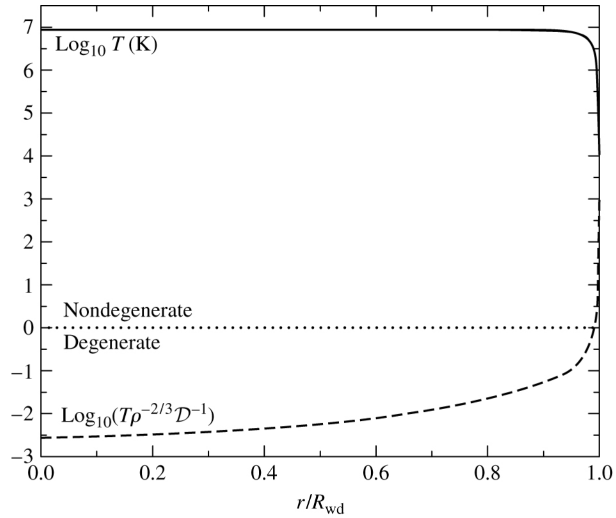

Figure 3: The temperature of WDs is isothermal out to the edge.

The time to cool is given by

$$-\frac{dU}{dt} = L \tag{26}$$

or

$$-\frac{d}{dt}\left(\frac{M}{Am_H}\frac{3}{2}kT_c\right) = CT_c^{7/2}\,, \tag{27}$$

which can be integrated to find

$$T_c(t) = T_0\left(1+\frac{5}{2}\frac{t}{\tau_0}\right)^{2/5}\,, \tag{28}$$

where τ_0 is the cooling timescale and T_0 is the initial temperature. We can use our expression for the luminosity to find

$$L(t) = L_0\left(1+\frac{5}{2}\frac{t}{\tau_0}\right)^{7/5}\,, \tag{29}$$

These equations tell us that the luminosity decays quickly at first, but slows its rate of change with time.