

ASTR 702

Gas Physics (Chapter 3)

Stars are gas! Specifically, they are ionized gas (except in their photospheres, which do contain some neutrals and not-completely-ionized species). We need to understand the properties of gas, the properties of radiation, and the interaction between gas and radiation.

1 The Equation of State

An “Equation of State” is a relation that relates temperature, pressure, and density - so-called “state variables:” $P = P(\rho, T, \vec{X})$. The most famous equation of state of course is the ideal gas law,

$$P = nkT. \quad (1)$$

It is vital that we understand how these state variables are related to each other.

We assume throughout this discussion that the gas is ideal, i.e. that thermal energy dominates over Coulomb energy. Is this a good assumption? Let’s see! The typical Coulomb energy per particle is:

$$\epsilon_C \approx \frac{1}{4\pi\epsilon_0} \frac{Z^2 e^2}{d}, \quad (2)$$

where Z_e is the charge per particle and d is the distance between particles. We can estimate d assuming a constant density:

$$d = \left(\frac{Am_H}{\bar{\rho}} \right)^{1/3} = \left(\frac{4\pi Am_H}{3M} \right)^{1/3} R, \quad (3)$$

where the mean particle mass is Am_H and the stellar radius and mass are R and M . Evaluating ϵ_C/kT , the ratio of Coulomb to kinetic energy per particle, we find ~ 0.01 . The ratio rises as Z increases, but doesn’t approach unity even for pure iron. As M decreases, however, the ratio does go up substantially. It only approaches unity for $M \lesssim M_\odot^{-3}$, which is why we must take Coulomb interactions into account for planets, but not stars.

1.1 Pressure

To derive an equation of state, we need a general relation for pressure. Free particles exert a pressure given by the “pressure integral:”

$$P = 1/3 \int v p n(p) dp, \quad (4)$$

where v is the particle velocity, p is its momentum and $n(p) dp$ is the number of particles per unit volume with momentum in the range p to $p + dp$. This famous equation is derived in the book (I won’t repeat it here). It is a general relation that we can use for all computations of pressure (gas/radiation/degenerate gas).

The total pressure within a star can be derived by evaluating the pressure integral for each gas species and adding the radiation pressure exerted by photons. Treating ions and electrons separately:

$$P = P_i + P_e + P_{\text{rad}} = P_{\text{gas}} + P_{\text{rad}}. \quad (5)$$

We often define β as the fractional gas pressure:

$$P_{\text{gas}} = \beta P \quad (6)$$

$$P_{\text{rad}} = (1 - \beta)P \quad (7)$$

1.1.1 Ion Pressure

The ion pressure is simply given by the ideal gas law:

$$P_i = n_i kT. \quad (8)$$

We can spend just a little time understanding n_i . To compute n_i , we must sum over all species:

$$n_i = \frac{\rho}{\mu_i m_h}, \quad (9)$$

where μ_i is the mean particle mass:

$$\frac{1}{\mu_i} \approx X + \frac{Y}{4} + \frac{1 - X - Y}{\langle A \rangle}, \quad (10)$$

where X is the fractional abundance of hydrogen, Y of helium, and $\langle A \rangle$ is the mean atomic mass of everything else (“metals”). For the Sun, $X = 0.707$, $Y = 0.274$, and $\langle A \rangle \approx 20$; this gives $\mu_i = 1.29$.

1.1.2 Electron Pressure

The electron pressure can be derived in much the same way. We can again start with the ideal gas law:

$$P_e = n_e kT. \quad (11)$$

and

$$n_e = \frac{\rho}{\mu_e m_h}. \quad (12)$$

Because each species contributes a different number of electrons per particle, we have to again sum over all species. If we assume full ionization, which is a fine assumption in stellar interiors (but not photospheres):

$$\frac{1}{\mu_e} \approx X + \frac{Y}{2}(1 - X - Y) \left\langle \frac{Z}{A} \right\rangle, \quad (13)$$

where $\langle \frac{Z}{A} \rangle$ is the average value for all metals; assuming this value is close to 2,

$$\frac{1}{\mu_e} \approx (1/2)(1 + X) \quad (14)$$

which yields $\mu_e \approx 1.17$ for the Sun.

The total gas pressure is then $P_{\text{gas}} = P_i + P_e = \left(\frac{1}{\mu_i} + \frac{1}{\mu_e}\right) k\rho T/m_H$. If we substitute

$$\frac{1}{\mu} = \frac{1}{\mu_i} + \frac{1}{\mu_e}, \quad (15)$$

we find $\mu = 0.61$ for the Sun. This tells us that for hydrogen, ion and electron pressure are equal and that for all heavier elements, electron pressure is higher.

1.1.3 Degeneracy and Relativistic Pressure

The above assumes that electrons are not degenerate and are not relativistic. But, as the pressure goes up, the Pauli exclusion principle comes into play. Pauli says that no two fermions can have the same set of quantum numbers. But, Pauli must be understood in tandem with the Heisenberg Uncertainty Principle:

$$\Delta V \Delta^3 p \geq h^3, \quad (16)$$

where V is the volume and p is the momentum vector. Thus

$$n(p)dp = 2/\Delta V = 2/h^3 4\pi p^2 dp \quad (17)$$

for $p < p_0$, where p_0 is the maximum momentum.

We can solve for p_0 :

$$n_e = \int_0^{p_0} n_e(p) dp, \quad (18)$$

which solves to

$$p_0 = \left(\frac{3h^2 n_e}{8\pi}\right)^{1/3} \quad (19)$$

Putting it all together:

$$P_{e,\text{deg}} = \frac{8\pi}{15m_e h^3} p_0^5 = \frac{h^2}{20m_e} \left(\frac{3}{\pi}\right)^{2/3} \frac{1}{m_e^{5/3}} \left(\frac{\rho}{\mu_e}\right)^{5/3} \quad (20)$$

and finally we have

$$P_{e,\text{deg}} = K_1' \left(\frac{\rho}{\mu_e}\right)^{5/3}, \quad (21)$$

with $K_1' = 1.00 \times 10^7 \text{ m}^4 \text{ kg}^{2/3} \text{ s}^{-2}$.

As $v \rightarrow c$, we have to treat things relativistically. We do the same treatment as above, but with $v = c$ and get

$$P_{e,r\text{-deg}} = \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} \frac{1}{m_H^{4/3}} \left(\frac{\rho}{\mu_e}\right)^{4/3} \quad (22)$$

and finally we have

$$P_{e,\text{deg}} = K_2' \left(\frac{\rho}{\mu_e} \right)^{4/3}, \quad (23)$$

with $K_2' = 1.24 \times 10^{10} \text{ m}^3 \text{ kg}^{-1/3} \text{ s}^{-1}$.

These are also equations of state! So we have three:

$$P = \frac{\rho}{m_g} kT \quad (24)$$

$$P_{e,\text{deg}} = K_1' \left(\frac{\rho}{\mu_e} \right)^{5/3} \quad (25)$$

$$P_{e,\text{deg}} = K_2' \left(\frac{\rho}{\mu_e} \right)^{4/3} \quad (26)$$

The big thing to notice is that for degenerate gas there is no temperature dependence.

We need two more pieces of information on the state of our gas: its ionization state and its electronic configuration. For the former, we'll use Saha and for the latter we'll use Boltzmann.

1.2 Ionization state

The Saha ionization equation relates the ionization state to the temperature and pressure.

For a gas composed of a single atomic species in LTE, only concerning two states (excited and not as excited) the Saha equation is written:

$$\frac{n_{i+1}n_e}{n_i} \simeq 2 \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} \frac{g_{i+1}}{g_i} \exp \left[-\frac{\Phi_r}{kT} \right], \quad (27)$$

where n_i is the density of atoms in the i -th state of ionization, that is with i electrons removed.

g_i is the degeneracy of states for the i -ions; note that this is not the electron degeneracy used in Boltzmann below!

Φ_r is the energy required to remove i electrons from a neutral atom, creating an i -level ion (the "ionization potential").

n_e is the electron density

m_e is the mass of an electron

T is the temperature of the gas

k_B is the Boltzmann constant

h is Planck's constant.

Hydrogen is particularly simple. The degeneracy of states for the ground state of hydrogen is 4 (proton spin up, electron up; p up e down; p down e up; p down e down). The degeneracy

of states for the ionized state is 2 (spin up and down). We therefore have:

$$\frac{n_{H^+}n_e}{n_H} \simeq \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \exp\left[-\frac{13.6 \text{ eV}}{k_B T}\right], \quad (28)$$

It's super useful to have the ionization potentials for common elements memorized because then one can get a rough sense for the ionization state of a gas. We can arrive at a very rough approximation by considering how tightly bound an electron is. For example, ionizing the second electron of He takes a lot of energy because the Coulomb interaction is strong.

1.3 Electronic state

We use the Boltzmann equation, the single most important equation in stat. mech. to tell us the electronic state:

$$\frac{n_i}{n_j} = \frac{g_i}{g_j} e^{-E_{ij}/k T_{\text{ex}}}, \quad (29)$$

where n_i is the density in state i , g_i is the degeneracy of state i , E_{ij} is the energy difference between the two states, and T_{ex} is the “excitation temperature.” More on T_{ex} later.

While this equation gives the relative densities between states, we are frequently interested in the fractional density of a given state compared to all states. In such cases, we need to use the “partition function,”

$$Z(T_{\text{ex}}) = \sum_j^{\infty} g_j e^{-E_j/k T_{\text{ex}}} \quad (30)$$

So that

$$\frac{n_i}{n_T} = \frac{g_i e^{-E_i/k T_{\text{ex}}}}{Z(T_{\text{ex}})}, \quad (31)$$

where n_T is the total population in all levels.

The excitation temperature is not a physical temperature! It is instead the temperature at which the Boltzmann equation is satisfied. When is $T_{\text{ex}} = T_k$ the kinetic temperature? When collisions are frequent! Assume we have two competing processes: collisions and radiation, and that the kinetic (collision) temperature T_k and radiation temperature (T_R) are different. If the timescale for collisions is closer than the timescale for photon-particle interactions, $T_{\text{ex}} \simeq T_k$.

For another example, assume that the population levels are inverted such that the upper level is overpopulated relative to the lower level, then T_{ex} is negative. This is allowed because T_{ex} is not a real temperature. These population inversions can result in masing emission.

It is also worth noting that the excitation temperature only corresponds to the transition between the upper and lower levels. Therefore, each transition can have a different excitation temperature!

Radiation temperature, T_R , is the the equivalent temperature blackbody that would emit the same intensity at the frequency of interest. This is also sometimes called the background temperature, T_{BG} or T_0 . In the limit of low frequencies where the RJ limit applies, the brightness and radiation temperatures are the same.

If $T_k = T_R = T_{ex} = T$, the system is in *thermodynamic equilibrium* (TE). This happens when the particle energy distribution follow the Boltzmann equation (and ionization states follow Saha, see below), the particle velocity distributions follow MB, and the radiation field is a Planck function at temperature T_R .

If $T_k = T_{ex} \neq T_R$, the system is in *local thermodynamic equilibrium*, LTE. LTE is much easier to attain and is commonly assumed. This is often good enough when collisions dominate over radiative processes. “Local” here refers to \sim one mean free path.

In LTE, the changes in temperature must vary slowly, so that at each point in the object of interest we can assume TE. That temperature is that of the particles, which follow a Maxwellian distribution with a single temperature, for all particle species. In other words, the temperature gradient scale must be small compared to the mean free path of the particles.

1.4 Radiation Pressure

Having done gas pressure in all its forms, we are ready to turn to radiation pressure. We can write the Planck function as

$$n(\nu)d\nu = \frac{8\pi\nu^2}{c^3} \frac{d\nu}{e^{h\nu/kT} - 1}, \quad (32)$$

where $n(\nu)d\nu$ is the number of photons with frequency between ν and $\nu + d\nu$. Using the pressure integral with $v = c$ and $p = h\nu/c$,

$$P_{\text{rad}} = 1/3 \int_0^\infty c \frac{h\nu}{c} n(\nu)d\nu, \quad (33)$$

and after inserting the Planck function and doing the integral we find

$$P_{\text{rad}} = 1/3aT^4, \quad (34)$$

where a is the radiation constant $a = 8\pi^5 k^4 / (15c^3 h^3) = 4\sigma/c$

Finally, we can put things in terms of energy per unit mass u . For non-degenerate, we have

$$u_{\text{gas}} = \frac{3}{2} \frac{P_{\text{gas}}}{\rho} = \frac{3}{2} \frac{nkT}{\rho} = \frac{3}{2} \frac{kT}{m_g}. \quad (35)$$

We arrive at exactly the same for generate! For relativistic, degenerate, we find

$$u_{\text{gas}} = 3 \frac{P_{\text{gas}}}{\rho}. \quad (36)$$

For radiation,

$$u_{\text{rad}} = \int_0^\infty h\nu n(\nu) d\nu = at^4 = 3 \frac{P_{\text{rad}} \rho}{\rho} \quad (37)$$

So, regardless of the degeneracy of the gas or even if we are discussing gas or radiation, we always have $u \propto P/\rho$. We will use this fact below.

2 The Adiabatic Exponent

We can often assume that stars are “adiabatic,” meaning within a parcel of gas there is no heat exchange with the outside environment. This assumption makes our life much easier. The adiabatic exponent, also known as the heat capacity ratio, tells us the relationship between pressure and density.

The big advantage of assuming adiabaticity is that we can apply the first law of thermodynamics:

$$du + Pd \left(\frac{1}{\rho} \right) = 0, \quad (38)$$

where du is the change in energy and $Pd(1/\rho)$ is the work done.

From the preceding discussion, we can write

$$u = \phi \frac{P}{\rho}, \quad (39)$$

where ϕ is a constant. We can compute du :

$$du = \phi \left[\frac{\rho dP - P d\rho}{\rho^2} \right] = \phi \left[\frac{dP}{\rho} - Pd \left(\frac{1}{\rho} \right) \right] \quad (40)$$

and therefore

$$(\phi + 1)Pd \left(\frac{1}{\rho} \right) + \phi \frac{1}{\rho} dP = 0 \quad (41)$$

or

$$\frac{P/dP}{\phi + 1} = \frac{\phi}{\rho} \frac{1}{d\rho}, \quad (42)$$

which is an ordinary linear differential equation and solves to

$$P = K_a \rho^{(\phi+1)/\phi} = K_a \rho^{\gamma_a}, \quad (43)$$

where γ_a is the adiabatic exponent and K_a is a proportionality constant that depends on the entropy of the system. Comparing with our earlier results, (for un-ionized atomic gas), $\gamma_a = 5/3$ for ideal and non-relativistic degenerate gas and $\gamma_a = 4/3$ for relativistic degenerate gas.

If we have ions, γ_a changes because ions affect the energy density. We then have an additional energy term representing the potential energy of ionization

$$\frac{\chi n_+}{\rho} = \frac{\chi n_+}{[(n_0 + n_+) m_H]} = \frac{\chi n_+}{m_H} \quad (44)$$

so therefore

$$u = 3/2 \frac{P}{\rho} + \frac{\chi n_+}{m_H}. \quad (45)$$

As before, we can compute du and substitute into the first law. Doing so, we get an incredibly ugly expression for γ_a :

$$\gamma_a(x) = \frac{5 + (5/2 + \chi/kT)^2 x(1-x)}{3 + [3/2 + (3/2 + \chi/kT)^2] x(1-x)}, \quad (46)$$

where x is the ionization fraction. Note that this does not apply to degenerate gas. We can see that if $x = 0$ or $x = 1$, $\gamma_a = 5/3$ as we found before. If $x = 0.5$, the function is at a minimum:

$$\gamma_a(x = 0.5) = \frac{5 + 0.5^2(5/2 + \chi/kT)^2}{3 + 0.5^2[3/2 + (3/2 + \chi/kT)^2]}, \quad (47)$$

which results in $\gamma_a = 1.63$ for $\chi/kT = 1$ and $\gamma_a = 1.21$ for $\chi/kT = 10$. The upshot is that γ_a is always near unity and varies between 1 and 2. Despite this, because it is in the exponent, these small changes can have a large impact.

We can arrive at approximate values for γ_a without going through all the math. If we assume energy equipartition, each degree of freedom adds $1/2kT$ to the energy. We can write the $\gamma \simeq (f + 2)/f$, where f is the number of degrees of freedom. So for monatomic gas, $f = 3$ (x, y, z) and $\gamma_a = 5/3$. For diatomic gas, $f = 5$ because there are two rotational degrees of freedom (it is symmetric under rotation of this third axis), so $\gamma_a = 7/5$.

3 Opacity

Opacity (κ) refers to the degree to which a material allows light to pass through it and is closely related to optical depth:

$$\tau_\nu = \int \kappa_\nu \rho ds, \quad (48)$$

where ρ is the mass density and the integration is over the path length. Opacity is intrinsic to the material, whereas optical depth is integrated along the path. We can see from the equation that opacity must have units of area/mass. Note that just like optical depth, there is a wavelength/frequency dependence to opacity.

We can think of opacity as the analogue of emission. High opacity means that the element is able to absorb at that frequency. The element therefore must be in the correct electronic state (if applicable), and the correct ionization state for that particular photon. The more of an element, the stronger the opacity. Therefore, $\kappa_\nu = \kappa_\nu(\rho, T, X_i)$.

Stellar opacity turns out to be incredibly important, as it is related to the luminosity. Opacity can remove photons from the medium (absorption) or simply redirect them and remove them from the line of sight (scattering).

There are five primary sources of opacity:

1. Bound-Bound, κ_{bb} . The photon excites an electron and the electron then relaxes back to the ground state, releasing photons. These are sharp features, because only certain photons can cause the transitions. If there are multiple steps between the excited and ground states, multiple photons can be released, each of less energy than the original. The net result will be a redening of the spectrum. So bound-bound has high opacity only at particular wavelengths, and requires neutrals (and so isn't important in stellar interiors).
2. Bound-free, κ_{bf} . These are photoionizing interactions. Any photon with energy greater than the ionization potential of the atom/molecule in question can photoionize it. Produces continuous opacity above the ionization potential.
3. Free-free absorption, κ_{ff} . Free electrons near ions can absorb photons over a continuous range of wavelengths. Important in stellar interiors.
4. Electron (Thompson) scattering, κ_{es} . Electrons can also scatter, or change the direction of, photons. This results in continuum opacity.
5. H^- Continuum Opacity, κ_{H^-} . It may seem strange, but hydrogen can capture an extra electron, leading to something called H^- . H^- only forms in relatively low temperatures, say for F0 stars and later. The ionization potential of H^- is just 0.754 eV, which means that a large number of photons can ionize it. It is a significant source of opacity in stellar photospheres (but is otherwise unimportant).

Each of these has its own characteristic functional form. Often we can model these functions as

$$\kappa = \kappa_0 \rho^a T^b, \quad (49)$$

where κ_0 is a constant that depends on the material, and a and b are constants that depend on the mechanism. One common form is when $a = 1$ and $b = -7/2$ - this form is known as “Kramer’s Opacity.” Let’s take each process in turn:

For bound-bound, we don’t have a functional form for opacity since the opacity depends strongly on the wavelength of the radiation and the composition of the material.

For bound-free radiation, the opacity is zero up to the ionization potential of a given element, then spikes and falls off. We can model the portion above the ionization potential:

$$\kappa_{\text{bf}} = \kappa_0 \frac{g_{\text{bf}}}{t} Z(1 + X) \rho T^{-7/2}, \quad (50)$$

where $\kappa_{\text{bf},0} = 4.32 \times 10^{25}$, g_{bf} is the Gaunt factor (approximately unity), and t is the “guillotine factor” that describes the atom’s contribution to the opacity after it has been ionized. This has a Kramer’s opacity dependence.

For free-free absorption, we have

$$\kappa_{\text{ff}} = \frac{\kappa_{\text{ff},0}}{\mu_e} \left\langle \frac{Z^2}{A} \right\rangle \rho T^{-7/2} \approx 1/2 \kappa_{\text{ff},0} (1 + X) \left\langle \frac{Z^2}{A} \right\rangle \rho T^{-7/2}, \quad (51)$$

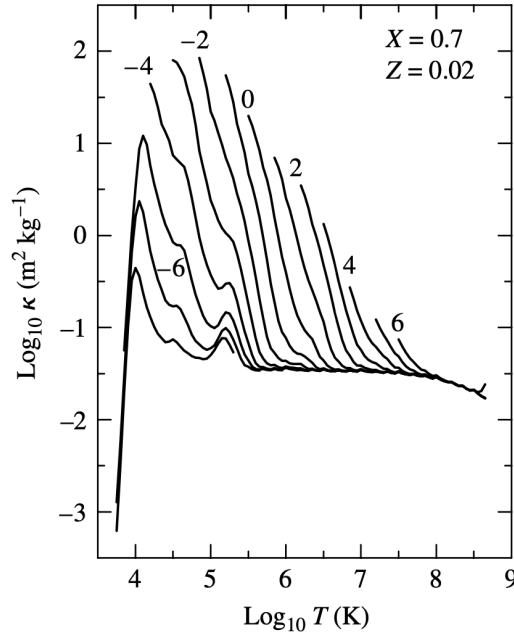


Figure 1: Computation of the Rosselland mean opacity. The curves are the logarithm of the density.

where in the approximation we have used the relations from earlier for μ_e ($1/\mu_e$ is the average number of free electrons per nucleon), $\kappa_{ff,0} = 7.5 \times 10^{18} \text{ m}^5 \text{ kg}^{-2} \text{ K}^{7/2}$. Our book lists that this is accurate to within 20%. This is also a Kramer's opacity term.

For electron scattering, there is no frequency or temperature dependence, $a = b = 0$. Thus

$$\kappa_{\text{es}} = \frac{\kappa_{\text{es},0}}{\mu_e} \approx 1/2 \kappa_{\text{es},0} (1 + X), \quad (52)$$

where $\kappa_{\text{es},0} = 0.04 \text{ m}^2 \text{ kg}^{-1}$. This is the dominant source of opacity in stellar interiors.

For H^- , C&O give a function form for $3000 \text{ K} < T < 6000 \text{ K}$ and $10^{-7} \text{ kg}^{-1} \text{ m}^{-3} < \rho < 10^{-2} \text{ kg}^{-1} \text{ m}^{-3}$:

$$\kappa_{\text{H}^-} \approx \kappa_{\text{H}^-,0} (Z/0.02) \rho^{1/2} T^9 \text{ m}^2 \text{ kg}^{-1}, \quad (53)$$

where $\kappa_{\text{H}^-,0} = 7.9 \times 10^{-34} \text{ m}^{-2} \text{ kg}^{-1}$. This is the dominant source of opacity in stellar photospheres.

We can define a Rosselland Mean Opacity to attempt to determine an opacity averaged over all wavelengths that depends only on the temperature:

$$\bar{\kappa} = \kappa_{\text{bb}} + \kappa_{\text{bf}} + \bar{\kappa}_{\text{ff}} + \kappa_{\text{es}} + \kappa_{\text{H}^-} \quad (54)$$

Because the sources of opacity are complicated, this is not a trivial calculation! One result is shown in the figure. Our book lists an average value of $\bar{\kappa} = 0.04 \text{ m}^2 \text{ kg}^{-1}$.

We typically see no further into a star (or anything optically thick) than unity optical depth. A more careful treatment actually shows that the level within a stellar atmosphere from which most of the photons of wavelength λ escape is at optical depth $\tau_\lambda \simeq 2/3$. Indeed, the condition $\tau_\lambda \simeq 2/3$ defines the stellar photosphere – the layer of a star’s atmosphere from which the light we see originates.

There are two consequences of this realization. First, the condition applies to all viewing angles; therefore, the distance ds corresponding to the condition $\tau_\lambda = 2/3$ will probe further into the star’s interior at the center of a stellar disk than at its edges. Second, recalling the definition of optical depth, it is obvious that if the opacity κ_λ increases at some wavelength, then ds must be smaller to satisfy the condition $\tau_\lambda = 2/3$. Thus, we see further into a star in its continuum light than at the wavelengths of discrete absorption lines. These two effects explain a phenomenon known as ‘limb darkening’, first recognised in the Sun, whereby the light emitted in successive annuli from the centre decreases in intensity and becomes progressively redder. Sightlines near the limb do not penetrate as deeply into the Sun’s atmosphere by the time $\tau_\lambda = 2/3$ is reached; since the Sun’s temperature decreases outwards from the center, such sightlines see light from cooler regions of the Sun’s atmosphere.

The opacity determines how energy is transferred through a star. It is therefore related to the temperature gradient. We will follow the treatment of Eddington, who equated the momentum of photons passing through a slab to the force of those photons. We can say that the former term is $\frac{dI}{c}$ (our book uses H), where I is the intensity $dI = \kappa\rho I ds$. Thus the momentum is $\kappa\rho I dr/c$ because $ds = dr$. The latter term is the pressure difference (these are both per unit area and a force is just pressure/area): $P(r)_{\text{rad}} - P(r + dr)_{\text{rad}}$. Thus

$$\frac{\kappa\rho I}{c} = -\frac{dP_{\text{rad}}}{dr}. \quad (55)$$

We know for Blackbody radiation that $P = 1/3aT^4$ and

$$I = -\frac{4acT^3}{3\kappa\rho} \frac{dT}{dr}. \quad (56)$$

We know $F = I \times \text{Area}$, so

$$F = -4\pi r^2 \frac{4acT^3}{3\kappa\rho} \frac{dT}{dr} \quad (57)$$

and we can invert this to

$$\frac{dT}{dr} = -\frac{3\kappa\rho}{4acT^3} \frac{F}{4\pi r^2} \quad (58)$$

or

$$\frac{dT}{dm} = -\frac{3\kappa}{4acT^3} \frac{F}{(4\pi r^2)^2} \quad (59)$$