ASTR702 - HW7
October 17, 2025, Due October 24, 2025
2 pt each part

1) Given the range of γ values required for dynamical stability, what is the range of polytropic indices that you would expect for stable stars?

2) (6 pt) We had in class three reasons why convection may dominate over radiation for energy transport. Re-read this section of the Big Orange Book to argue for why the interior of massive stars, the outer layer of Solar mass stars, and the entirety of low mass stars is convective.

3) Adiabatic Index as a Function of Temperature During Ionization

The adiabatic index (or heat capacity ratio) $\gamma = C_p/C_v \simeq (f+2)/f$, where f is the number of degrees of freedom, describes how a gas responds to adiabatic compression or expansion. For an ideal monatomic gas at constant composition, $\gamma = 5/3$. However, when a gas undergoes ionization, the number of degrees of freedom changes, causing γ to vary with temperature.

Consider a partially ionized hydrogen plasma. As temperature increases, the ionization fraction α (defined as the ratio of ionized hydrogen to total hydrogen nuclei) increases from 0 to 1.

a) Write the total number of particles N in terms of ionization α. Verify that this works for full and no ionization.

b) Write the total internal energy, which has contributions from the translational kinetic energy of all particles $\frac{3}{2}k_BT$ per particle (in 3D) and the ionization energy per ionized particle (which depends on the ionization potential χ). The total internal energy U is a function of T, α, N, and χ.

c) Find the heat capacity at constant volume, $C_V = (\partial U/\partial T)_V$. Note that α itself depends on temperature through the Saha equation.

d) Calculate $\gamma = C_P/C_V$. I'll give you C_P (although you can calculate it from the first law of thermodynamics):

$$C_P = C_V + Nk_B\left(1+\alpha+T\frac{d\alpha}{dT}\right)$$

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e) What is γ for fully neutral, fully ionized, and partially ionized gas? Why is the partially ionized value different?

f) (4 pt) The Saha equation relates ionization fraction to temperature. We can write it in terms of α:

$$\frac{\alpha^2}{1-\alpha} = \frac{2}{n_e}\left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} \exp\left(\frac{\chi}{k_B T}\right)$$

where n_e is the electron density. In the case of partial ionization far from complete ionization (i.e., $\alpha \ll 1$), we can show that:

$$\frac{d\alpha}{dT} \approx \frac{\alpha}{T}\left(\frac{\chi}{2k_B T} + \frac{3}{2}\right)$$

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Substitute this result back into your expression for γ and discuss how γ varies with temperature during the ionization phase. Sketch (or describe) the behavior of $\gamma(T)$ as temperature increases through the ionization zone.

g) Explain why the adiabatic index increases during ionization using C_P and C_V.

h) (Grad students only, 6 pt) Now imagine you have a star that has H and He. Draw a qualitative graph of γ vs T. Assume that H ionized near $T \sim 10000$ K, He near $T \sim 20000$ K and the second He ionization is near $T \sim 40000$ K.

i) (Grad students only, 4 pt) Compute the temperature above which partially ionized hydrogen will always be dynamically stable. In your solution, first create a plot of $\gamma_\alpha(T)$ versus T (sketch is fine) to prove that they are inversely proportional. Then solve for T by setting $\gamma_\alpha = 4/3$, our stability criterion. What is your interpretation of the meaning of this temperature?