GR and Black Holes
C+O Chapter 17

The final endpoint of stellar evolution for us to discuss is black holes, which are the products of the most massive stars created. The treatment of black holes necessitates a discussion of general relativity, GR.

GR

In 1915, Einstein published his new theory of gravity, GR. At the time, Newtonian gravity was well-accepted, but there were hints that other physics remained unaccounted for. Foremost was the orbit of Mercury, for which the perihelion location (furthest distance from the Sun) shifts in a manner that cannot be explained by Newtonian gravity. Einstein in 1905 published his theory of special relativity, which reconciles Newton's laws of motion with electrodynamics. A new theory was needed to update gravity.

Equivalence principle

Special relativity tells us that the laws of physics are the same in all inertial reference frames. What is an inertial reference frame? A frame that is not accelerating! But, since gravity is an acceleration, and everything acts under the influence of gravity, can we really have inertial frames? We know that we do have inertial frames, so although gravity causes an acceleration, it must be different.

[from Wikipeda] An observer in an accelerated reference frame must introduce what physicists call fictitious forces to account for the acceleration experienced by himself and objects around him. One example is the force pressing the driver of an accelerating car into his or her seat; another is the force you can feel pulling your arms up and out if you attempt to spin around like a top. Einstein's master insight was that the constant, familiar pull of the Earth's gravitational field is fundamentally the same as these fictitious forces. The apparent magnitude of the fictitious forces always appears to be proportional to the mass of any object on which they act - for instance, the driver's seat exerts just enough force to accelerate the driver at the same rate as the car. By analogy, Einstein proposed that an object in a gravitational field should feel a gravitational force proportional to its mass, as embodied in Newton's law of gravitation.

A person in a free-falling elevator experiences weightlessness; objects either float motionless or drift at constant speed. Since everything in the elevator is falling together, no gravitational effect can be observed. In this way, the experiences of an observer in free fall are indistinguishable from those of an observer in deep space, far from any significant source of gravity. Such observers are the privileged ("inertial") observers Einstein described in his theory of special relativity: observers for whom light travels along straight lines at constant speed.

Einstein hypothesized that the similar experiences of weightless observers and inertial observers in special relativity represented a fundamental property of gravity, and he made this the cornerstone of his theory of general relativity, formalized in his equivalence principle. Roughly speaking, the "equivalence principle" states that a person in a free-falling elevator cannot tell that they are in free fall. Every experiment in such a free-falling environment has the same results as it would for an observer at rest or moving uniformly in deep space, far from all sources of gravity.

We can see how odd this is by equating Newton's second law with the gravitational force:

$$ma = \frac{GMm}{r^2} \tag{1}$$

But the LHS shows an object's resistance to acceleration (its "inertial" mass) and the RHS gives the gravitational force, with m and M being the "gravitational charges." By why would these two masses be the same? Think of the situation for EM:

$$ma = \frac{qQ}{4\pi\epsilon_0 r^2}. \tag{2}$$

Strictly, we should change our nomenclature

$$m_i a = \frac{GM_g m_g}{r^2} \tag{3}$$

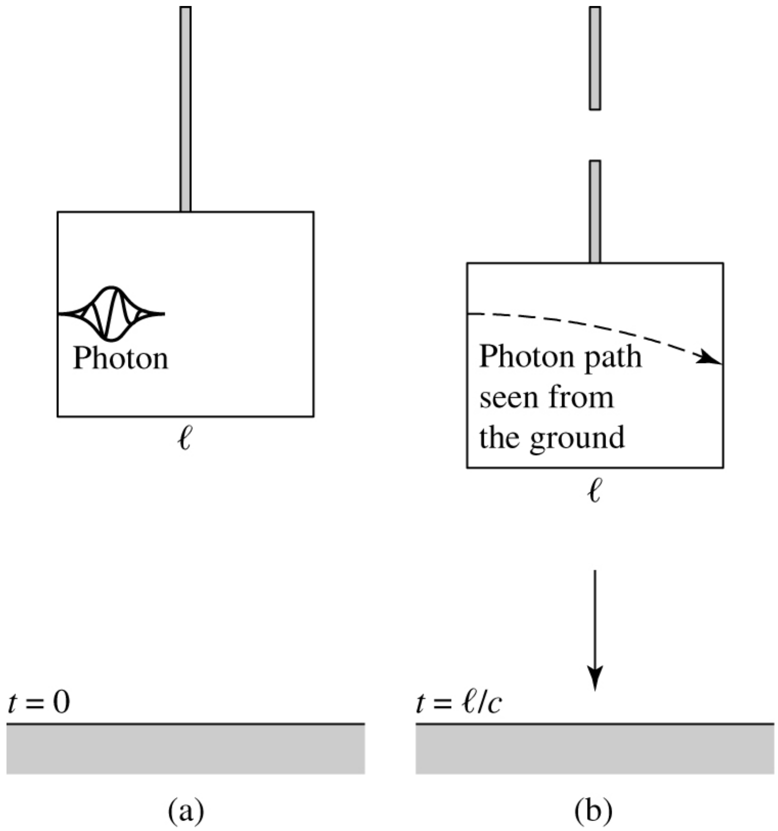
Experiments have shown that m_i/m_g is unity to within one part in 10^{12}. This is known as the "weak equivalence principle."

Gravity behaves differently. Einstein realized that if everyone were in a state of free-fall, we would have no method of knowing it! Also, since each spatial location feels a different force of gravity, it cannot be fully eliminated. Instead, Einstein proposed "local" reference frames where the acceleration due to gravity is essentially constant, which is known as the "Principle of Equivalence:" all local, free-falling nonrotating laboratories are fully equivalent for the performance of all physical experiments.

Curved Light

Let's do another thought experiment! Imagine you have suspended a laboratory high above the earth. From one side of the lab you shoot a photon horizontal to the ground while at the same time releasing the lab so it is in free-fall. In the reference frame of the lab, the photon must maintain its horizontal displacement. An observer located on earth, however, would see the photon's path bend downward toward the Earth as the lab fell.

This deflection is minor, but measurable. The deflection of the photon is the quickest path to the other side of the lab, and evidence for curved spacetime.



(4)

Redshift

Redshift is a concept in astrophysics that describes the shifting of the frequency of light. We can define the redshift z as

$$z = \frac{\Delta\lambda}{\lambda_0} = \frac{\Delta\nu}{\nu_0}\,. \tag{5}$$

This is the definition, but we can relate it to the relative velocity via the Doppler effect:

$$z = \sqrt{\frac{1 + v_r/c}{1 - v_r/c}} - 1 \tag{6}$$

As long as $v_r \ll c$, we can make the expression

$$(1 + v_r/c)^{\pm 1/2} \simeq 1 \pm \frac{v_r}{2c}\,, \tag{7}$$

so for nonrelativistic motions

$$z = \frac{\Delta\lambda}{\lambda_0} \simeq \frac{v_r}{c} \tag{8}$$

By convention, we call decreases in wavelength (increases in frequency) blueshifts, for when the source of radiation is moving toward the observer. We call increases in wavelength (decreases in frequency) redshifts, for when the source of radiation is moving away from the observer.

Gravitational Redshift and Time Dilation

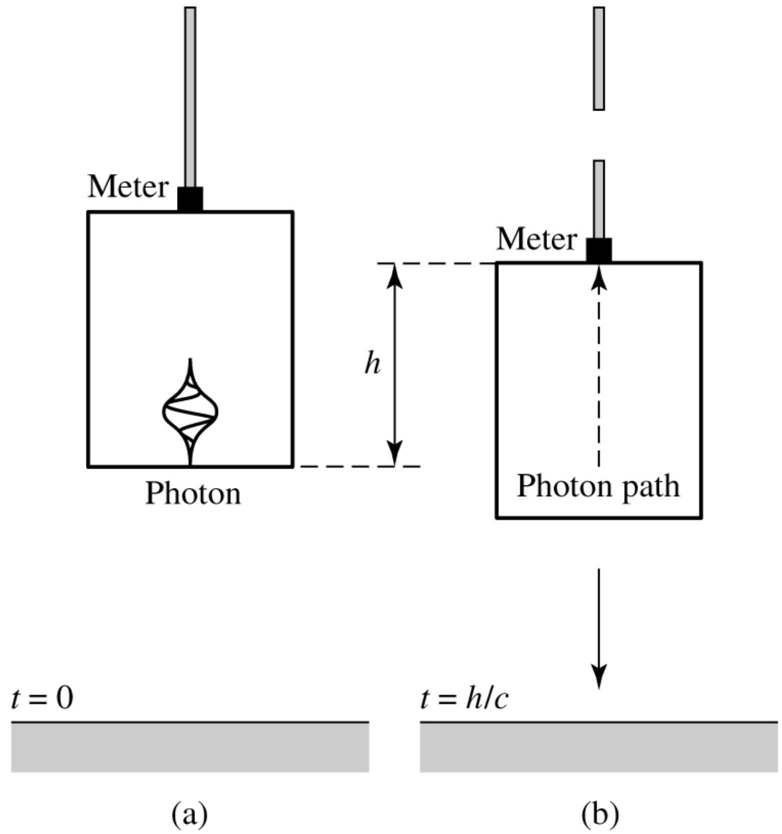
Let's again suspend a lab above the ground and cut the cable holding it just as we release a photon. This time, let's release the photon straight up from the lab's floor. Again, observers in the lab would not sense anything different with regards to the photon's motion.

From an external perspective, if the lab fell by distance h during the photon's travel, the ceiling is h closer to the photon than it was at the beginning. We would expect to measure a blueshift

$$\frac{\Delta\nu}{\nu_0} = \frac{v}{c} = \frac{gh}{c^2}\,. \tag{9}$$

But we know that such a shift is not measured. There must be something to oppose it, a "gravitational redshift" that applies when viewing accelerated frames. These frames can simply be due to gravity. Gravitational redshift is given by

$$\frac{\Delta\nu}{\nu_0} = -\frac{v}{c} = -\frac{gh}{c^2}\,. \tag{10}$$



(11)

The total gravitational redshift is given by integrating Equation 10 from r_0 to infinity. We must use $g = GM/r^2$ and set $h = dr$. A word of caution on this integration: the integration adds up contributions from different reference frames, but Equation 10 was derived from a local reference frame. The integration is only valid if spacetime is relatively flat.

$$\int_{\nu_0}^{\nu_\infty} \frac{d\nu}{\nu} \simeq -\int_{r_0}^{\infty} \frac{GM}{r^2 c^2} dr\,, \tag{12}$$

which results in

$$\ln\left(\frac{\nu_\infty}{\nu_0}\right) \simeq -\frac{GM}{r_0 c^2} \tag{13}$$

which is valid for relatively weak gravity and can be rewritten as

$$\frac{\nu_\infty}{\nu_0} \simeq e^{-GM/r_0 c^2}\,. \tag{14}$$

$e^{-x} \simeq 1 - x$ if $x \ll 1$ so

$$\frac{\nu_\infty}{\nu_0} \simeq 1 - \frac{GM}{r_0 c^2} \tag{15}$$

A more accurate derivation gives us

$$\frac{\nu_\infty}{\nu_0} \simeq \left(1 - \frac{2GM}{r_0 c^2}\right)^{1/2}\,. \tag{16}$$

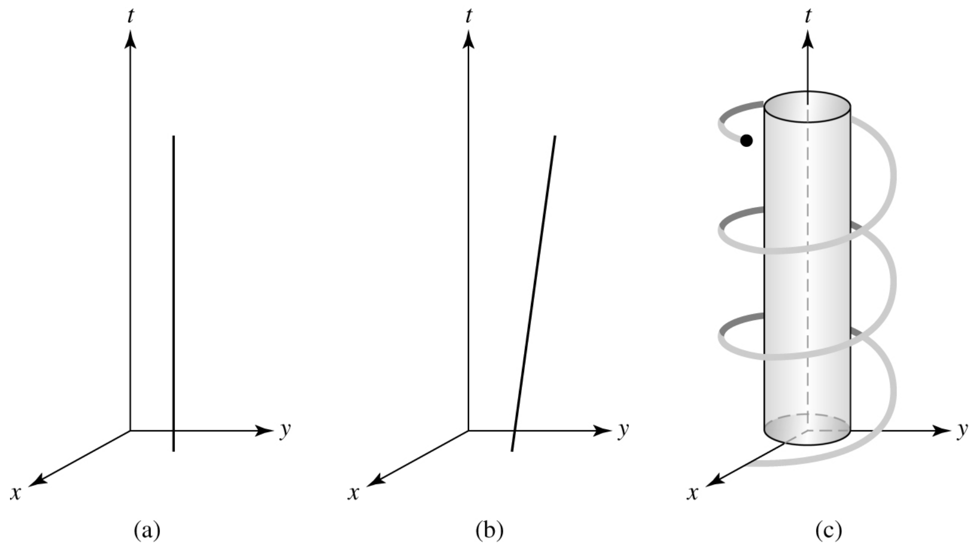


Figure 1: Worldlines!

Let's put things in terms of redshift z:

$$z = \frac{\Delta\nu}{\nu_0} = \frac{\nu_0}{\nu_\infty} - 1 \tag{17}$$

$$= \left(1 - \frac{2GM}{r_0 c^2}\right)^{-1/2} - 1 \tag{18}$$

$$\simeq \frac{GM}{r_0 c^2} \tag{19}$$

Intervals and Geodesics

The heart of GR is Einstein's field equations. The derivation and application of these equations is beyond the scope of this course (and way beyond my understanding!). These equations relate the effect of mass (and energy!) on the curvature of spacetime. This is the basic tenet of GR: mass and energy curve spacetime.

Spacetime diagrams can help us to get an intuitive feel for these equations. On these diagrams, worldlines show the path of an object: straight up for stationary objects, diagonal for constant velocity, more complicated for more complicated motion (but always moving upwards).

Light obviously moves at a constant velocity, and so should move as a straight line in a spacetime diagram. We can define a "lightcone" emanating from a single point in spacetime that gives all possible spacetime paths for a photon. Running time backwards gives all past possible paths. This leaves wide areas of spacetime that are not accessible! The finite speed of light means that not all of spacetime is accessible.

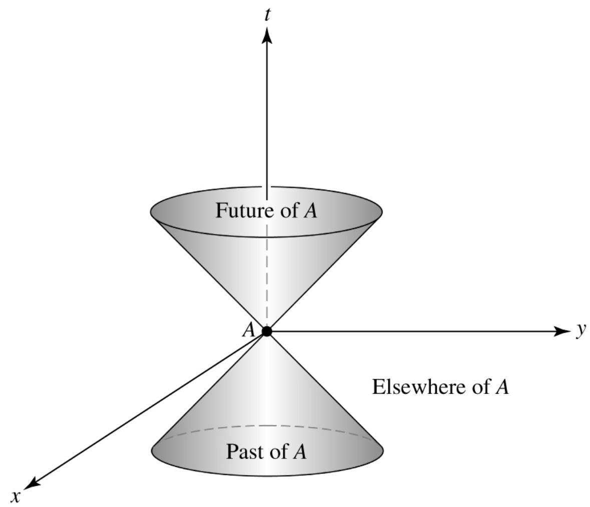


Figure 2: Light cones for photons.

Spacetime Intervals

We need to define a spacetime "distance." For cartesian distances of points specified as (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$(\Delta\ell)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \tag{20}$$

But in spacetime we have one more dimension, leading to (x_A, y_A, z_A, t_A) and (x_B, y_B, z_B, t_B):

$$(\Delta s)^2 = [c(t_B - t_a)]^2 - (x_B - x_A)^2 - (y_B - y_A)^2 - (z_B - z_A)^2\,, \tag{21}$$

or the squared interval is the distance traveled by light squared minus the distance between events squared.

Note that $(\Delta s)^2$ can be positive, negative, or zero. If positive, the interval is timelike and light has enough time to travel between events A and B. We can then choose an inertial reference frame such that A and B happen at the same spatial location. Because the two events occur at the same place, the time measured between the two events is $\Delta s/c$. By definition, the time between the two events that occur at the same location is the proper time

$$\Delta\tau = \frac{\Delta s}{c} \tag{22}$$

The proper time is the elapsed time recorded by a watch moving along the worldline from A to B.

If $(\Delta s)^2 = 0$, we call it lightlike or "null." In this case, only light can go from A to B.

If $(\Delta s)^2 < 0$, the interval is spacelike and light cannot make the travel. Thus, nothing can make the travel. The lack of simultaneity in this situation means that there are inertial frames in which the time-ordering of the events is reversed, or where they occur at the same time. In the frame where the two events occur at the same time, we can define a proper distance:

$$\Delta L = \sqrt{-(\Delta s)^2}\,. \tag{23}$$

If a straight rod were connected between A and B, this would be the rest length of the rod.

We can define a "metric" as the differential distance along any (possibly curved) path:

$$(d\ell)^2 = (dx)^2 + (dy)^2 + (dz)^2 \tag{24}$$

Light will always follow the shortest possible path, but this need not be a straight line. We can integrate the above equation to find $\Delta\ell$.

Two events can be connected by infinitely many curved worldlines. We can then define a metric for flat spacetime:

$$(ds)^2 = (cdt)^2 - (d\ell)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{25}$$

As before, we can integrate this to find the total interval.

In flat spacetime, a straight timelike worldline between two events is a maximum; any other wordline between the same two events will not be straight and will have a smaller interval.

Geodesics

In a non-flat spacetime (one with mass), the straightest possible worldlines are curved. These are called "geodesics." The paths followed by freely falling objects are geodesics.

In curved spacetime, a geodesic is an extremum. For this chapter, the geodesics we will encounter are maxuma.

Our overall goal here is to describe spacetime around a massive, spherical object. For this application, spherical coordinates are more useful:

$$(d\ell)^2 = (dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2 \tag{26}$$
$$(ds)^2 = (cdt)^2 - (dr)^2 - (rd\theta)^2 - (r\sin\theta d\phi)^2\,, \tag{27}$$

where the above expression is for flat spacetime. We need one that includes curvature. Note that the coordinates used here are those of an observer at rest a great ($\sim$ infinite) distance from the origin.

The derivation of the curved spacetime metric is way beyond the scope of this course. Karl Schwartzschild first solved Einstein's equations to get

$$(ds)^2 = \left(cdt\sqrt{1 - 2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1 - 2GM/rc^2}}\right)^2 - (rd\theta)^2 - (r\sin\theta d\phi)^2\,, \tag{28}$$

Although the derivation of the Schwartzschild metric is beyond our reach, we can see that it has the expected properties. If we set $dt = 0$, a distance along a radial line (with $d\theta = d\phi = 0$) is the proper distance:

$$dL = \sqrt{-(ds)^2} = \frac{dr}{\sqrt{1 - 2GM/rc^2}} \,. \tag{29}$$

This tells us that the spatial distance dL is greater than the coordinate difference dr. This is the stretching of spacetime around massive objects. We can do the same thing with proper time when r is unchanging:

$$d\tau = \frac{ds}{c} = dt\sqrt{1 - \frac{2GM}{rc^2}} \,. \tag{30}$$

Since $d\tau < dt$, time passes more slowly close to a massive object, as measured by an external observer.

The Orbit of a Satellite

Mass tells spacetime how to curve and spacetime tells mass how to move. Masses will follow the straightest possible worldline.

Let's say the satellite travels with $\omega = v/r$ entirely in the $\hat{\phi}$ direction, so $dr = d\theta = 0$ and $d\phi = \omega dt$. We can input these into the Schwartzschild metric to find

$$(ds)^2 = \left[\left(cdt\sqrt{1 - 2GM/rc^2}\right)^2 - r^2\omega^2\right] dt^2 \tag{31}$$

$$= \left(c^2 - \frac{2GM}{c} - r^2\omega^2\right) dt^2 \tag{32}$$

We can integrate this expression over one orbit to get

$$\Delta s = \int_0^{2\pi\omega} \sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\, dt \tag{33}$$

The worldline must have a radial derivative of zero, so

$$\frac{d}{dr}(\Delta s) = \frac{d}{dr}\int_0^{2\pi\omega} \sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\, dt = 0 \tag{34}$$

$$\frac{d}{dr}\sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2 dt} \tag{35}$$

$$\frac{2GM}{r^2} - 2r^2\omega^2 = 0 \tag{36}$$

$$v = r\omega = \sqrt{\frac{2GM}{r}} \tag{37}$$

This is the coordinate speed of the satellite for a circular orbit (the speed measured by a distant observer). Note that this is exactly what Newtonian gravity would predict!

The Schwartzschild Radius

On your homework, you will compute the Schwartzschild radius by equating the kinetic and potential energy of a photon. This (incorrect) derivation for the event horizon nevertheless gets the correct answer. We can prove this by taking a null worldline, $ds = 0$ for a photon falling straight into a black hole. Such a photon has $d\theta = d\phi = 0$, so

$$\frac{dr}{dt} = c\left(1 - \frac{2GM}{rc^2}\right) = c\left(1 - \frac{R_S}{r}\right) . \tag{38}$$

We see that the coordinate speed equals c at great distance, but as $r \to R_S$, $dr/dt \to 0$. We cannot get light inside the Schwartzschild radius. This is also called the 'event horizon." Interior to this is the "singularity." Light is frozen at the event horizon. In fact the collapse of the star is frozen there too! We cannot see it because photons are trapped, but if they weren't we would see time standing still.

A Trip into a Black Hole

What happens as one falls into a black hole? Assume you take a trip to a black hole while shining a flashlight backwards from your direction of travel. Light from that flashlight will be more and more redshifted to an external observer as you call in. It will also grow dimmer to this same observer as time dilation increases the time between photons. But to you, all is well, at least for a while! Eventually, the gravitational force on your feet is much greater than that on your head (a tidal force). This is bad. You are eventually stretched out like spaghetti.

Hawking Radiation

It may feel like black holes are impossible forces, given that they are the end point of many stars' evolutionary paths. Stephen Hawking discovered, however, that black holes do "'evaporate" over long timescales. We know from quantum mechanics that the quantum world is filled with particles popping in an out of existence. This produces pairs of particles and their anti-equivalents. When these pairs recombine, they annihilate and all is well. Near a black hole however, one of the particles may fall into the event horizon, and the other may escape the system. This carries energy away from the black hole, leading to its evaporation.

The timescale is really long here!

$$t_{\rm evap} = 2650\pi^2 \left(\frac{2GM}{c^2}\right)^2 \left(\frac{M}{M_\odot}\right) \approx 2\times 10^{67} \left(\frac{M}{M_\odot}\right) \,{\rm yr} . \tag{39}$$

So normal black holes take a long time! Smaller black holes formed in the big bang primordial may be as small as 10^{-8} kg according to your book, and so may be evaporating now. There is no evidence that this is the case, however.