ASTR 7~~05~~

HW #2

1a) $\frac{n_u}{n_e} = \frac{g_u}{g_e} e^{-h\nu_{ue}/kT_s} = 3 e^{-h\nu_{ue}/kT_s}$

b) Lets look at the exponent

$\nu_{ue} = 1420$ MHz $(=21 cm)$

$\Rightarrow \frac{h\nu}{k} = 0.07$ K

$n(HI) = n_u + n_e = n_u + \frac{n_u}{3} e^{h\nu_{ue}/kT_s}$

$= 4/3\, n_u\, e^{\frac{0.07}{T_s}}$

If $T_s = 100$ K, $n(HI) = 4/3\, n_u\, e^{0.07/100}$

$= 1.0007 \cdot 4/3\, n_u$

c) If $T_s = 20$ K, $n(HI) = 1.003 \cdot 4/3\, n_u$

2) Derive $e^{-h\nu/kT_{ex}} = \dfrac{e^{-h\nu/kT_k} C_{ul} + \dfrac{1}{e^{h\nu/kT_R}-1} A_{ul}}{C_{ul} + \left(\dfrac{1}{e^{h\nu/kT_R}-1}+1\right) A_{ul}}$

From detailed balance,

$$n_l (C_{lu} + B_{lu} u_\nu) = n_u (C_{ul} + B_{ul} u_\nu + A_{ul})$$

From the hint, assume collisions dominate, because those relations must always hold.

Neglect Einstein coefficients

$$n_l C_{lu} = n_u C_{ul}$$

but

$$\frac{n_u}{n_l} = \frac{g_u}{g_l} e^{-h\nu/kT_{ex}}$$

We know that in high densities when collisions dominate $T_{ex} \to T_k$, so

① $$\frac{C_{lu}}{C_{ul}} = \frac{g_u}{g_l} e^{-h\nu/kT_k}$$

Also,

$$A_{ul} = \frac{8\pi h\nu^3}{c^3} B_{ul} \Rightarrow B_{ul} = \frac{c^3}{8\pi h\nu^3} A_{ul} \quad ②$$

③ $$g_u B_{ul} = g_l B_{lu}$$

$$u_\nu = \frac{4\pi}{c} \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT_R}-1} = \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT_R}-1}$$

Rearranging detailed balance

$$\frac{n_u}{n_l} = \frac{C_{lu} + B_{lu} u_\nu}{C_{ul} + B_{ul} u_\nu + A_{ul}}$$

Plug in ① + ③

$$\frac{n_u}{n_l} = \frac{C_{ul} \frac{g_u}{g_l} e^{-h\nu/kT_K} + \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{-h\nu/kT_R} - 1} \frac{g_u}{g_l} B_{ul}}{C_{ul} + \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{-h\nu/kT_R} - 1} B_{ul} + A_{ul}}$$

Plug in ②

$$\frac{n_u}{n_l} = \frac{C_{ul} \frac{g_u}{g_l} e^{-h\nu/kT_K} + \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{h\nu/kT_R} - 1} \frac{c^3}{8\pi h\nu^3} \frac{g_u}{g_l} A_{ul}}{C_{ul} + \frac{8\pi h\nu^3}{c^3} \frac{1}{e^{-h\nu/kT_R} - 1} \frac{c^3}{8\pi h\nu^3} \frac{g_u}{g_l} A_{ul} + A_{ul}}$$

$$= \frac{g_u}{g_l} \frac{C_{ul} e^{-h\nu/kT_K} + \frac{1}{e^{-h\nu/kT_R} - 1} A_{ul}}{C_{ul} + \frac{1}{e^{h\nu/kT_R} - 1} A_{ul} + A_{ul}}$$

$$\Rightarrow e^{-h\nu/kT_{ex}} = \frac{e^{-h\nu/kT_K} C_{ul} + \frac{1}{e^{-h\nu/kT_R} - 1} A_{ul}}{C_{ul} + \left(\frac{1}{e^{-h\nu/kT_R} - 1} + 1 \right) A_{ul}}$$

3a) $I_\nu = (I_{\nu,\,source} + I_{\nu,\,BG})\, e^{-\tau_\nu} + S_\nu (1 - e^{-\tau_\nu})$

$F_\nu = \Omega I_\nu$

b) Case 1:

$I_{\nu,W} = (I_{\nu,\,source} + I_{\nu,\,BG})\, e^{-\tau_W} + B_\nu(T_W)(1 - e^{-\tau_W})$

→ for passing through warm slab only. I assumed LTE ($S_\nu \to B_\nu(T)$)

$$I_\nu = I_{\nu,w}\, e^{-\tau_c} + B_\nu(T_c)(1 - e^{-\tau_c})$$

$$= (I_{\nu,\,source} + I_{\nu,\,BG})\, e^{-(\tau_W + \tau_c)} + B_\nu(T_W)(1 - e^{-\tau_W})\, e^{-\tau_c}$$

$$+ B_\nu(T_c)(1 - e^{-\tau_c})$$

Flux would just be above expression times Ω.

Case 2 is just Case 1 but with $\tau_c \leftrightarrow \tau_w$ and $T_c \leftrightarrow T_w$

c) $I_\nu = I_{\nu,\,BG}\, e^{-(\tau_W + \tau_c)} + B_\nu(T_W)(1 - e^{-\tau_W})\, e^{-\tau_c}$

$+ B_\nu(T_c)(1 - e^{-\tau_c})$

for Case 1

Flux again $I\Omega$

$$\Delta I = I^{on} - I^{off}$$

$$= \left(I_{\nu, \text{source}} + I_{\nu BG}\right) e^{-(\tau_w + \tau_c)} - I_{\nu, BG} \left(e^{-(\tau_w + \tau_c)}\right)$$

$$= I_{\nu, \text{source}} \, e^{-(\tau_w + \tau_c)}$$

$$\Rightarrow \quad \tau_w + \tau_c = -\ln\left(\frac{\Delta I}{I_{\nu, \text{source}}}\right)$$