ASTR 367 — Stellar Interiors
C+O Chapter 10

We cannot directly observe stellar interiors, but must instead use observations of stellar "surfaces" to infer what is going on inside.

There are four equations of stellar structure. This lecture mainly concerns deriving these four fundamental equations.

Hydrostatic Equilibrium

Stars emit energy via fusion, which causes a pressure, or force, outward from their cores (where fusion takes place). Gravity tries to contract stars, and so provides a pressure or force inwards. These forces must be balanced in most stars, or else the stars would change in size. The balance of these two forces is called "hydrostatic equilibrium," and it is one of the most important topics in understanding stars.

The equation of hydrostatic equilibrium can be derived following C+O, but I won't repeat that derivation here. The result is

$$\frac{dP}{dr} = -G\frac{M_r \rho(r)}{r^2} = -\rho g\,, \tag{1}$$

where dP/dr is the radial change in the pressure outward from the core, M_r is the mass **interior** to radius r, $\rho(r)$ is the mass density **at** radius r, and $g = GM_r/r^2$.

Equation 1 assumes that the star is stable, not contracting or expanding, which is the case for stars on the main sequence. If the internal pressure is too great, the condition of hydrostatic equilibrium fails and the star must expand or contract. We will return to this point when discussing stellar evolution. This is the first equation of stellar interiors.

This equation states that in order to balance gravity, there must be a pressure gradient! The minus sign shows that this gradient is such that pressure is highest at low r (in the star's core) and lowest at high r (at the star's surface).

Mass Conservation

It is frequently useful to know how M_r varies with radius. We can logic our way to an expression of mass conservation by assuming a thin spherical shell of mass dM_r and thickness dr. If $dr \ll r$, the volume is $dV = 4\pi r^2 dr$, and we can write

$$dM_r = (4\pi r^2 dr)\rho(r)\,, \tag{2}$$

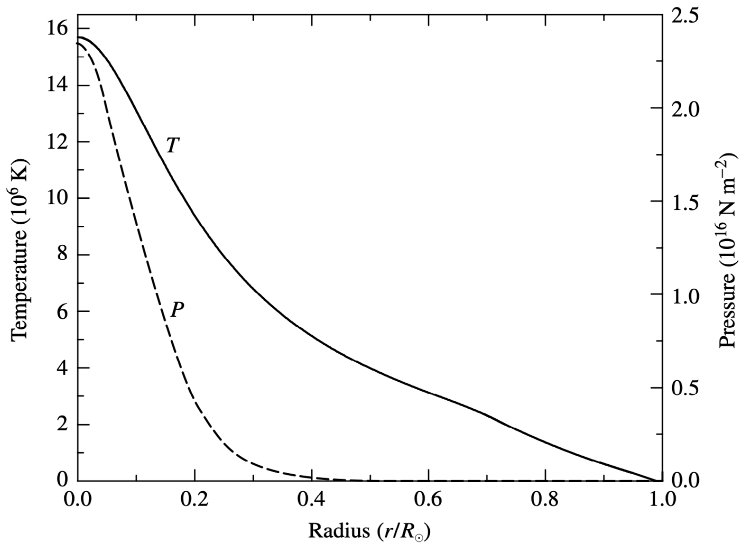

Figure 1: Temperature and pressure within the Sun.

or

$$\frac{dM_r}{dr} = 4\pi r^2 \rho(r)\,, \tag{3}$$

which is the mass conservation equation. Given $\rho(r)$, we can integrate this equation over all r to get the star's mass. This is the second equation of stellar interiors.

Luminosity Equation

The luminosity equation relates the luminosity gradient to the energy production rate. Since luminosity is just energy per time, these quantities are natural to use.

If we have infinitesimal mass dm, the luminosity $dL = \epsilon dm$, where ϵ is the total energy released per kilogram per second. For a star, $dm = dM_r = \rho(r)dV = 4\pi r^2 \rho(r) dr$. Therefore,

$$\frac{dL_r}{dr} = 4\pi r^2 \rho(r)\epsilon\,. \tag{4}$$

This is our third equation of stellar interiors.

The Temperature Gradient

OK, so those three laws were not so tricky to derive. No such luck going forward, though. The temperature gradient equation will require some work.

Part of the issue is that there are multiple ways for energy to escape a star. Let's review!

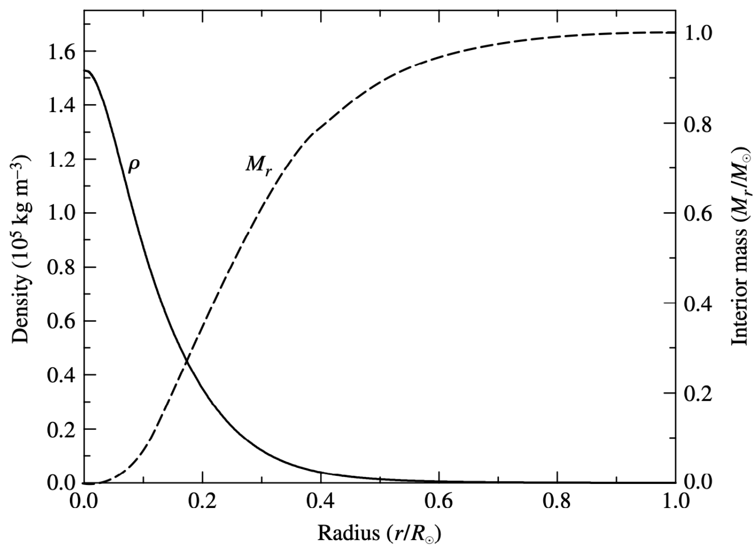

Figure 2: Mass and density within the Sun.

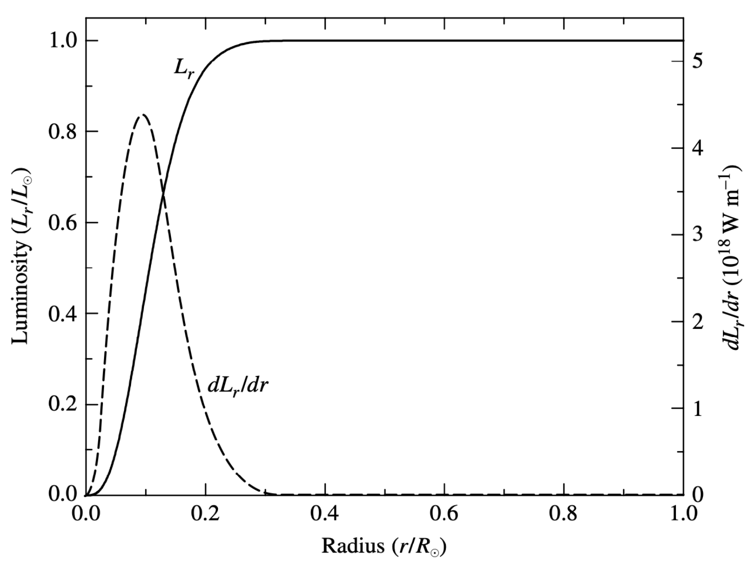

Figure 3: Luminosity within the Sun.

We know energy is produced in stellar interiors and gets out of the Sun. There are three mechanisms: radiation (emission of photons), convection (bouyant higher temperature gas moving toward the stellar surface), and conduction (particles exchanging energy). Conduction is not as important in stars, so we'll focus just on the first two mechanisms.

Equation of State

Before we go further, we need to dive into the stellar equation of state. Equations of state are equations that tell us how pressure, volume, temperature, and/or energy are related.

The most familiar equation of state is the ideal gas law:

$$P = nkT\,, \tag{5}$$

where n is the number density.

We can derive more general equations of state using the **pressure integral**:

$$P_{\rm gas} = \frac{1}{3}\int_0^\infty n_p p v dp\,, \tag{6}$$

where P is the pressure p is the momentum, n_p is the number of particles of momentum p, and v is the velocity. This equation looks confusing, but if we simply take the momentum $p = mv$ and use the Maxwell-Boltzmann speed distribution, we derive the ideal gas law.

We frequently want a different form of the ideal gas law, and so use the "mean molecular weight":

$$\mu = \frac{\bar{m}}{m_H}\,, \tag{7}$$

where m_H is the mass of a hydrogen atom. The mean molecular weight is a misnomer, since the particles can be atoms or ions too and we are concerned with mass, not weight - it's really just the mean mass per particle.

Rewriting the ideal gas law we get

$$P_{\rm gas} = \frac{\rho kT}{\mu m_H}\,. \tag{8}$$

What are some common values for μ? Values less than 1 are for ionized gas and those greater than 2 are for molecules. This is the relevant equation of state for gas in a star.

Radiation pressure is also at play. For radiation pressure, the photons do not follow a Maxwell-Boltzmann distribution. Instead, $n_p dp = n_\nu d\nu$, which gives (after a few lines):

$$P_{\rm rad} = \frac{1}{3}aT^4\,, \tag{9}$$

where a is the radiation constant. This is the relevent equation of state for photons in a star.

Radiative Temperature Gradient

Radiation Pressure

Problem 9.16 in C+O has you solve for the radial pressure gradient

$$\frac{dP_{\rm rad}}{dr} = -\frac{\bar{\kappa}\rho(r)}{c} F_{\rm rad} \,, \tag{10}$$

where $\bar{\kappa}$ is the mean opacity and $F_{\rm rad}$ is the force of radiation pressure. If we differentiate the radiation pressure of Equation 9 by dr,

$$\frac{dP_{\rm rad}}{dr} = \frac{4}{3} a T^3 \frac{dT}{dr} \,, \tag{11}$$

and note that $F = L/(4\pi r^2)$ we arrive at the temperature gradient for radiative transport:

$$\frac{dT}{dr}_{\rm radiative} = -\frac{3}{4ac} \frac{\bar{\kappa}\rho(r)}{T^3} \frac{L(r)}{4\pi r^2} \,. \tag{12}$$

This expression says that the if the flux or opacity increases, the temperature gradient must become steeper (more negative) if radiation is to transport the required luminosity outward. This also applies to density increases or temperature decreases.

This is half-way there, but we also have to deal with convection.

Eddington Luminosity

But first, a quick aside... Very massive stars have extreme energy generation rates and extreme radiation pressures. Since $F_{\rm rad} = L(r)/4\pi r^2$, we can write

$$\frac{dP}{dr} \simeq -\frac{\kappa\rho(r)}{c} \frac{L(r)}{4\pi r^2} \,. \tag{13}$$

But hydrostatic equilibrium says

$$\frac{dP}{dr} = -G \frac{M\rho(r)}{r^2} \,. \tag{14}$$

Putting these together, we have

$$L_{\rm Ed} = \frac{4\pi G c}{\bar{\kappa}} M \,. \tag{15}$$

This is the "Eddington Luminosity", the maximum luminosity a star can attain. It also sets the maximum mass, around 100 $M_\odot$.

Convection

Convection is much more complicated, but we can arrive at some insight if we assume **adiabatic** convection. Adiabatic means that no heat flows into or out of the gas. In this case, we have the adiabatic gas law:

$$PV^{\gamma} = K \, , \tag{16}$$

where $P_{\rm gas}$ is the pressure, K is a constant, V is the volume, and γ is the "adiabatic index" or heat capacity ratio. Common values for γ are $\gamma = 5/3 = 1.\overline{666}$ for monotonic gas, $\gamma = 7/5 = 1.4$ for diatomic gas, and $\gamma = 4/3 = 1.\overline{333}$ for a plasma. The adiabatic index tells you how much heat is required to raise the temperature; lower values of γ mean that more heat is required.

If we then differentiate the ideal gas law,

$$\frac{dP_{\rm gas}}{dr} = -\frac{P_{\rm gas}}{\mu}\frac{d\mu}{dr} + \frac{P_{\rm gas}}{\rho}\frac{d\rho}{dr} + \frac{P_{\rm gas}}{T}\frac{dT}{dr} \tag{17}$$

and also differentiate the adiabatic gas law:

$$\frac{dP}{dr} = \gamma\frac{P}{\rho}\frac{d\rho}{dr} \tag{18}$$

If we further assume that μ is a constant, we can combine the above two equations with the equation of hydrostatic equilibrium and simplify to obtain

$$\frac{dT}{dr}_{\rm adiabatic} = -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2} \tag{19}$$

Radiation vs. Convection

Stars have radiative zones and convective zones. As we'll see later, only low mass stars are fully convective. The Sun has a radiative zone in the interior and a convective zone outside of that. High mass stars have the opposite: radiative zones in their interiors and convective zones outside.

Your book goes into a long derivation to determine when convection dominates over radiation. The upshot is that convection dominates when

$$\frac{d\ln P}{d\ln T} < \frac{\gamma}{\gamma - 1} \, . \tag{20}$$

Although this equation is difficult to parse, we can say where convection will dominate: 1) in regions of high opacity where dT/dr would be too large for radiative transport, 2) where ionization is occurring, which causes a large specific heat and low temperature gradiant dT/dr, or 3) where the temperature dependence on nuclear energy generation is large, so dF/dr and dT/dr are large.

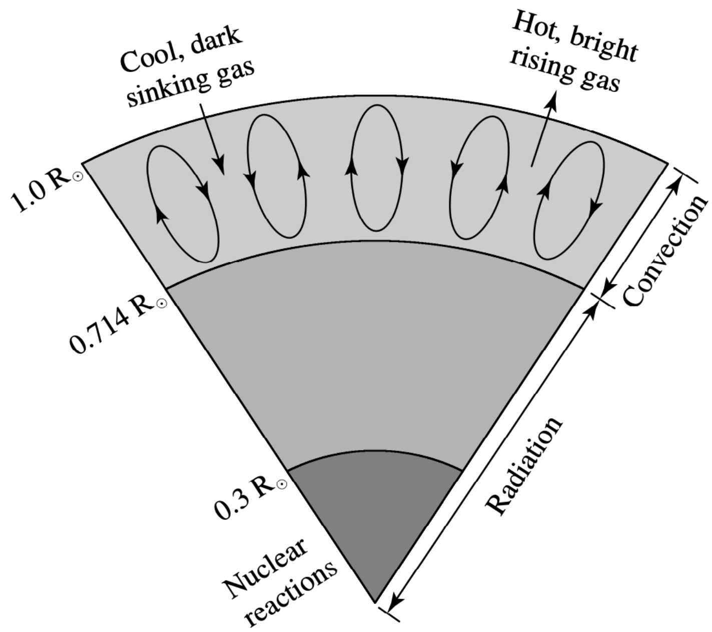

Figure 4: Schematic of zones within the Sun.

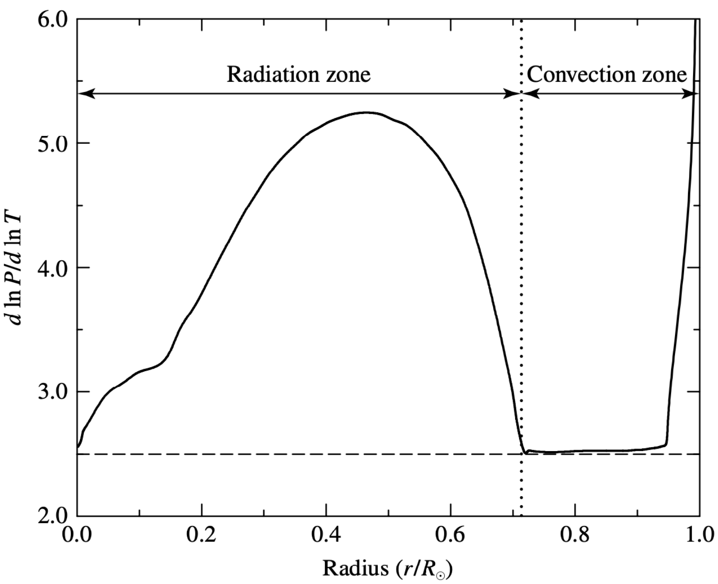

Figure 5: Criterion for radiation/convection zones within the Sun.

The Four equations of Stellar Structure

So, finally we can write our four equations:

$$\frac{dP}{dr} = -G\frac{M_r \rho(r)}{r^2} = -\rho(r) g \,, \tag{21}$$

$$\frac{dM_r}{dr} = 4\pi r^2 \rho(r) \,, \tag{22}$$

$$\frac{dL}{dr} = 4\pi r^2 \rho(r) \epsilon(r) \,., \tag{23}$$

$$\begin{aligned} \frac{dT}{dr}_{\text{radiative}} &= -\frac{3}{4ac}\frac{\bar{\kappa}\rho(r)}{T^3}\frac{L(r)}{4\pi r^2} \\ \frac{dT}{dr}_{\text{adiabatic}} &= -\left(1 - \frac{1}{\gamma}\right)\frac{\mu m_H}{k}\frac{GM_r}{r^2} \end{aligned} \tag{24}$$

Togeter, these equations allow us to model energy production in stars.

Stellar Models

To construct a model star, we must simultaneously satisfy all four equations of stellar structure. Because physical quantities change as a function of r, these equations must be evaluated in "shells," or in 1D.

The boundary conditions are that $M_r \rightarrow 0$ and $L(r) \rightarrow 0$ as $r \rightarrow 0$. Also, $T, P, \rho \rightarrow 0$ as $r \rightarrow R_*$.

When is convection important? Lower mass stars are fully convective. This is important! All the byproducts of fusion can be brought to the surface via convection. High mass stars are not! For the Sun, convection dominates between $0.7 R_\odot$ and the photosphere, whereas radiation dominates interior to $0.7_\odot$.

The change from radiative to convective happens because the Sun becomes opaque - radiation has trouble escaping.

As we've said before, stellar mass is the only important parameter in determining a star's properties and subsequent evolution (the book phrases it differently, but this is the gist of it).

Nucleosynthesis

Although we have now described the equations that govern stellar structure, we haven't talked about how stars actually make energy. We'll focus here on "normal" stars that make

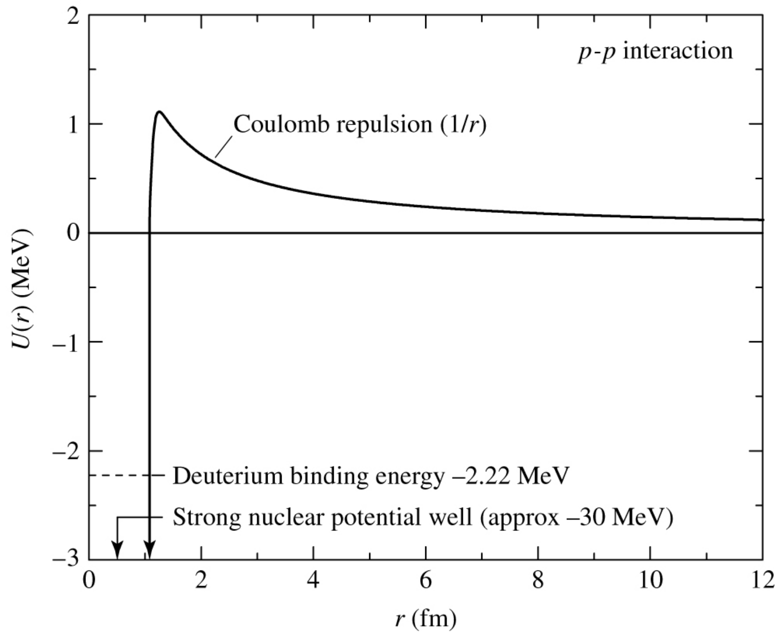

Figure 6: Potential energy for nuclear reactions.

energy via fusion, as opposited to white dwarfs or neutron stars.

Nucleosynthesis is the process by which stars create energy via fusion. In the normal main sequence portion of a star's life, it fuses four hydrogen nuclei (protons) into one helium nucleus (an alpha particle). This process has many steps, and can take many forms. Below I'll describe common pathways for the fusion reactions.

Reaction Rates

Stars create energy via nuclear fusion. To fuse together, elements must overcome the Coulomb repulsion. If the Coulomb force can be overcome, then the strong force takes over and fusion can begin. For proton-proton reactions, Figure 10.4 in your book gives the relevant graph.

How do we get atoms close enough together? Collide them quickly! This is why fusion only happens at high temperature. Atoms need sufficient kinetic energy to overcome the tunnel through the Coulomb barrier. The derivation of the reaction rate is rather complex, but we can determine the relevant terms.

To find the reaction rate in # reactions per unit volume per unit time, we can consider the number of particles hitting cross sectional area σ. So the number of incident particles within a cylindrical volume is

$$dN_E = \sigma(E) v(E) n_{iE} dE dt \,. \tag{25}$$

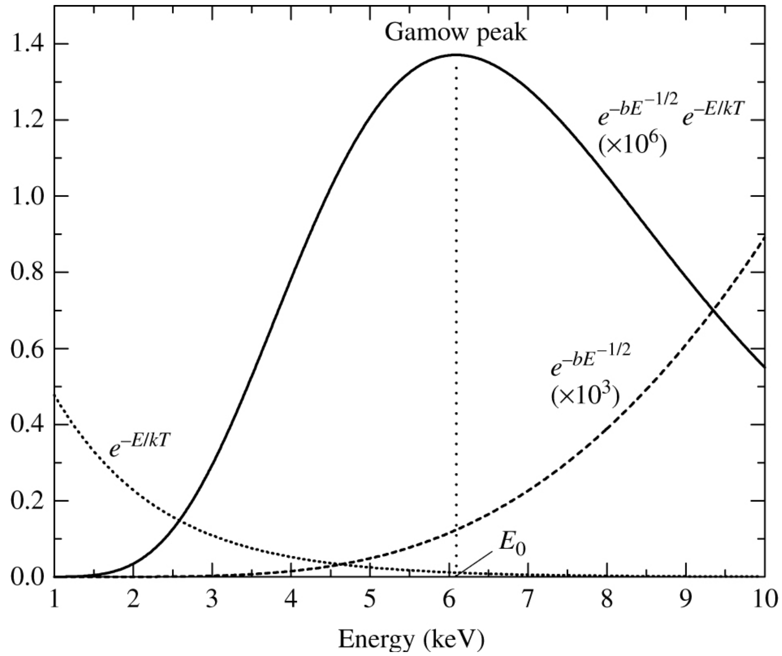

Figure 7: Nuclear reaction probability for proton-proton.

Only particles with the correct energy matter, or $n_{iE} dE = \frac{n_i}{n} n_E dE$ so we can write

$$\frac{\text{reactions per nucleus}}{\text{time interval}} = \sigma(E) v(E) \frac{n_e}{n} n_E dE\,. \tag{26}$$

Deriving these terms gets a bit ugly, so I will say without proof that $\sigma(E) \propto d^{E^{-1/2}}$ and from M-B, $v(E) \propto e^{-E/kT}$. Multiplying these two terms together results in a peak probability at a particular energy. This peak probability is called the "Gamow peak" after the physicist who first derived it.

Proton-Proton Chain

The proton-proton chain is the most common fusion sequence in most stars.

$$4{}^1_1\text{H} \rightarrow {}^4_2\,\text{He} + 2e^+ + 2\nu_e + 2\gamma \tag{27}$$

For one chain, the reactions are:

$${}^1_1\text{H} + {}^1_1\,\text{H} \rightarrow {}^2_1\,\text{H} + e^+ + \nu_e \tag{28}$$
$${}^2_1\text{H} + {}^1_1\,\text{H} \rightarrow {}^3_2\,\text{He} + \gamma \tag{29}$$
$${}^3_2\text{He} + {}^3_2\,\text{He} \rightarrow {}^4_2\,\text{He} + 2{}^1_1\text{H} \tag{30}$$

The first reaction is the slowest. This set of three equations is known as PPI, the most common reaction to fuse hydrogen into helium. There are., however, other ways to fuse H

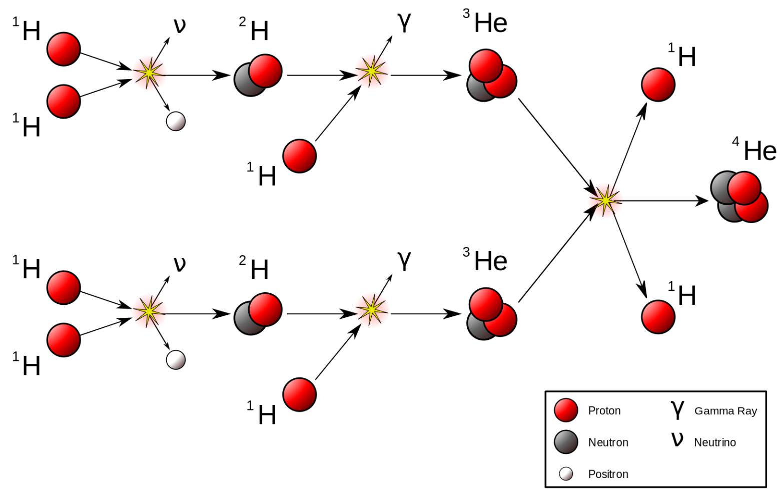

Figure 8: The PPI chain.

into He. PPII and PPIII are separate chains that use the first steps of PPI; they are listed in the book. These three chains together are known as the proton-proton chain.

Note the production of neutrinos (ν_e)! These will be important later.

The energy generated through the PP chain scales as temperature to the fourth power, when the temperature is around 1.5×10^7 K.

CNO Cycle

Larger stars can use the CNO cycle, in addition to PP. CNO stands for carbon, nitrogen and oxygen, but *these are just catalysts!* No C,N, or O are produced in the CNO cycle. Hydrogen is still being converted into helium, just through a different method.

The CNO cycle is *strongly* temperature dependent, the emitted energy scales as temperature to the 19.9 power! This means that at higher core temperatures the CNO cycle takes over energy production.

PP vs CNO

PP and CNO are the main reactions that power stars when they are on the main sequence.

The reactions of the CNO cycle require more kinetic energy to overcome the Coulomb barrier. Additionally, the CNO cycle's temperature dependence is much stronger. As a result, small stars get essentially all their energy from PP, whereas large stars get essentially all of theirs from CNO. Stellar mass stars get about half their energy from each.

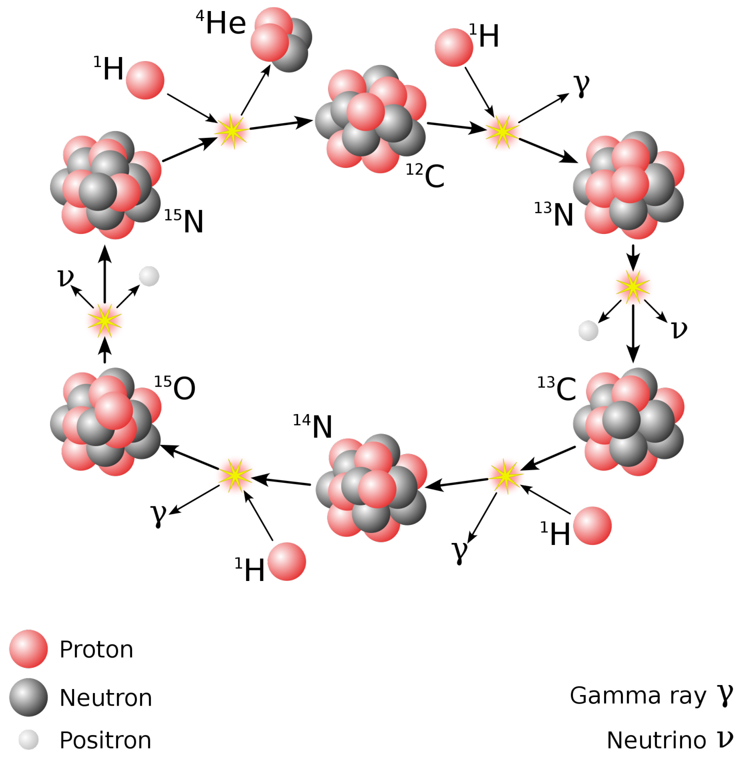

Figure 9: The CNO cycle.

Triple Alpha

During fusion, mass is converted into energy. The star's mass actually decreases, gravity is less powerful, and the pressure decreases so the star remains in hydrostatic equilibrium. Thus, as a star ages, different fusion processes may become available.

The triple alpha process combines three helium nuclei (alpha particles) to create carbon. This does not happen on the main sequence, only later in a star's life. It requires very high temperatures, and has a temperature dependence to the 41st power! That's insane! Combining two alpha particles makes a beryllium nucleus, which will decay rapidly if not struck by a third alpha particle; thus high temperatures are required for the reactions to proceed.

The End of the Road

There are yet more processes available at the end of a star's life, fusing more and more massive elements.

Reactions release energy (exothermic) until iron is produced. Iron is the peak of the binding energy curve, which means that it requires a lot of energy to change its configuration. Re-

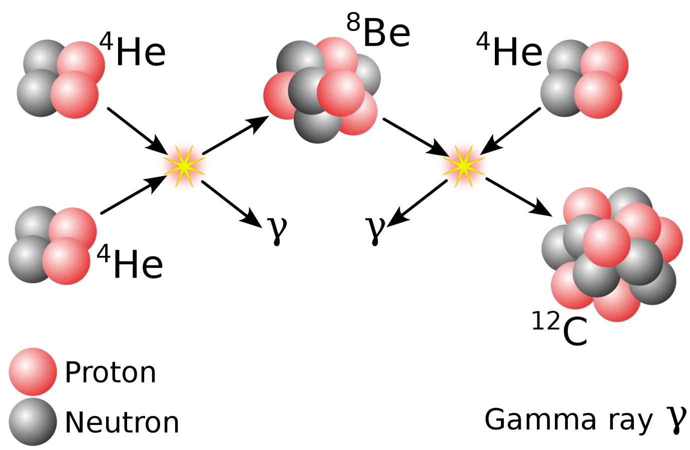

Figure 10: Priple-alpha process.

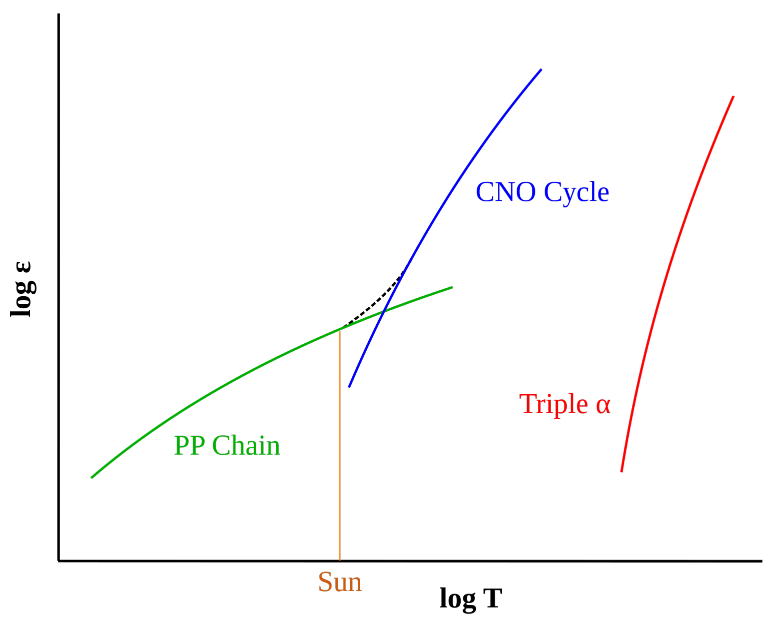

Figure 11: Nuclear energy generation.

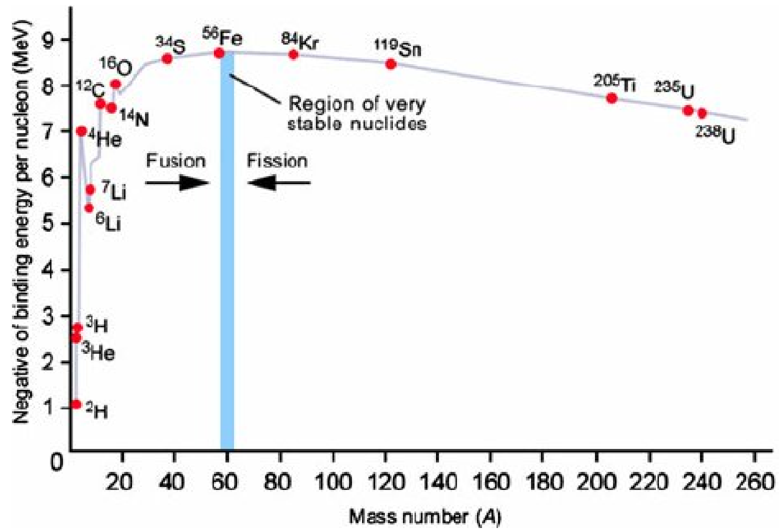

Figure 12: The binding energy curve. Elements to the left of the line release energy via fusion; those to the right release energy via fission.

actions making elements more massive that iron require energy (endothermic). The upshot is that fusion only produces elements up to iron.