ASTR 702

HW #2

1a) $\rho(r) = \rho_c[1 - r/R]$

$$\int_0^M dm = \int_0^R 4\pi r^2 \rho(r)\, dr = \int_0^R 4\pi r^2 \rho_c [1 - r/R]\, dr$$

$$= 4\pi\rho_c \int_0^R \left[r^2 - \frac{r^3}{R}\right] dr$$

$$\frac{M}{4\pi\rho_c} = \frac{R^3}{3} - \frac{R^3}{4} = \frac{R^3}{12}$$

$$\Rightarrow \rho_c = \frac{12M}{4\pi R^3} = \frac{3M}{\pi R^3}$$

1b) $\frac{dP}{dr} = -\frac{Gm}{4\pi r^4}$

$$\int_{P_c}^{P} dP = \int_0^{r'} \frac{G m(r) \rho(r)}{r^2}\, dr$$

$$m(r) = \int_0^m dm = \int_0^R 4\pi r^2 \rho(r)\, dr = 4\pi \rho_c \left(\frac{r^3}{3} - \frac{r^4}{4R}\right)$$

$$= 4\pi \rho_c r^3 \left(\frac{4R - 3r}{12R}\right) = \frac{\pi \rho_c r^3}{3R}(4R - 3r)$$

$$P - P_c = \int_0^r \frac{G}{r^2} \underbrace{\frac{\pi \rho_c r^3}{3R}(4R - 3r)}_{m(r)} \underbrace{\rho_c (1 - r/R)}_{\rho(r)}\, dr$$

integrates to

$$= \frac{G\pi \rho_c^2 r^2}{36R^2}\left(9r^2 - 28rR + 24R^2\right)$$

$$P = P_c - G\pi \rho_c^2 r^2 \left(\frac{r^2}{4R^2} - \frac{7r}{9R} + \frac{2}{3}\right)$$

for $r = R$, $P = 0$, so

$$P_c = G\pi \rho_c^2 R^2 \left(\frac{1}{4} - \frac{7}{9} + \frac{2}{3}\right) = \frac{5 G\pi \rho_c^2 R^3}{36}$$

but $\rho_c = \dfrac{3M}{\pi R^3}$

$$\Rightarrow P_c = \frac{5\pi G R^2}{36} \cdot \frac{9M^2}{\pi^2 R^6}$$

$$= \frac{5}{4\pi} \frac{GM^2}{R^4}$$

$$P_c = \frac{5}{4\pi}\frac{GM^2}{R^4} = \frac{5G}{4\pi}\frac{M_\odot^2}{R_\odot^4}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R}{R_\odot}\right)^{-4}$$

$$= 4.4\times 10^{14}\left(\frac{M}{M_\odot}\right)^2\left(\frac{R}{R_\odot}\right)^4 \ N\,m^{-2}$$

1c) $P = nkT = \dfrac{\rho}{\bar{m}_g} kT$

$$P_c = \frac{\rho_c}{\bar{m}_g} kT_c$$

$$\Rightarrow T_c = \frac{P_c \bar{m}_g}{k \rho_c}$$

$$P_c = \frac{5}{4\pi} \frac{GM^2}{R^4} \quad , \quad \rho_c = \frac{3M}{\pi R^3}$$

$$T_c = \frac{5}{4\cancel{\pi}} \frac{GM^2}{R^4} \frac{\cancel{\pi} \cancel{R^3} \bar{m}_g}{3\cancel{M}} = \frac{5}{12} \frac{GMm_g}{kR}$$

1d) $\Omega = -\int_0^M \frac{G m(r)}{r}\, dm$

$m(r) = \frac{\pi \rho_c r^3}{3R}(4R - 3r)$ (from 1b)

$\Omega = -G \int_0^M \frac{\pi \rho_c r^3}{3R}(4R-3r)\, dm$

$dm = 4\pi r^2 \rho_c (1 - r/R)\, dr$

$\Omega = -\frac{4\pi^2 G \rho_c^2}{3R} \underbrace{\int_0^R r^4 (4R-3r)(1 - r/R)\, dr}_{\text{integrates to } \frac{13}{210} R^6}$

$\Omega = \frac{4\cancel{\pi^2} G}{3R} \cdot \left(\frac{3M}{\cancel{\pi R^3}}\right)^2 \cdot \frac{13}{210} \cancel{R^6} = \frac{4 \cdot 13 \cdot 9\, G M^2}{3R \cdot 210}$

$= \frac{4 \cdot 13 \cdot 3}{210} \frac{GM^2}{R} = \frac{156}{210} \frac{GM^2}{R} = 0.74 \frac{GM^2}{R}$

Virial: $2U + \Omega = 0$

$U = \int_0^M u\, dm = \frac{3}{2} \int_0^M \frac{P}{\rho}\, dm = \frac{3}{2} \int_0^R \frac{P}{\rho} \cdot 4\pi r^2 \rho(r)\, dr$

$$= 6\pi \int_0^R P r^2 \, dr$$

from (b), $P = P_c - \frac{G\pi \rho_c^2 r^2}{36R^2}(9r^2 - 28rR + 24R^2)$

$$= 6\pi \int_0^R \underbrace{P_c r^2 - \frac{G\pi \rho_c^2 r^4}{36R^2}(9r^2 - 28rR + 24R^2)}\, dr$$

$$= 2\pi P_c R^3 - \frac{6\pi \cdot 149\pi G \rho_c^2 R^5}{3780}$$

$\rho_c = \frac{3M}{\pi R^3}$ $\qquad P_c = \frac{5}{4\pi} \frac{GM^2}{R^4}$

$$= 2\cancel{\pi} \cdot \frac{5}{4\cancel{\pi}} \frac{GM^2}{R} - \frac{6 \cdot \cancel{\pi^2} \cdot 149 \cdot 9}{3780 \cdot \cancel{\pi^2}} \frac{GM^2}{R}$$

$$= \left(\frac{5}{2} - \frac{447}{210}\right)\frac{GM^2}{R} = \frac{78}{210}\frac{GM^2}{R} = \tfrac{1}{2}\Omega \quad \checkmark$$

2) $\tau = \int \kappa \rho \, ds$

if $\kappa, \rho = \text{const.}$

$\tau = \kappa \rho s$

$$s = \frac{\tau}{\kappa \rho} = \frac{2/3}{0.3\ \text{cm}^2/\text{g} \cdot 2\times10^{-7}\ \text{g}/\text{cm}^3} = 1.1\times10^7\ \text{cm}$$