ASTR 702
Stellar Equations (Chapter 2

1 Local Thermodynamic Equilibrium

We often have to assume that stellar properties change slowly with respect to the average interaction time between particles. The condition is known as "Local Thermodynamic Equilibrium" (LTE) and means that we can determine the structure of a star given its density, temperature, and composition.

If we go to the extreme and assume that there is no temperature change at all, we have the condition of thermodynamic equilibrium (TE) at temperature T. There is no change in intensity along the path and $\frac{dI_\nu}{d\tau_\nu} = 0$. In this case, $I_\nu = S_\nu = B_\nu(T)$, our old friend the Planck function. When is $I_\nu = B_\nu(T)$??? When $d\tau \rightarrow \infty$! Or in other words, when the optical depth is high, the intensity is that of a blackbody at temperature T. In this case, nothing else about the source matters, only its temperature.

In LTE, the changes in temperature must vary slowly, so that at each point in the object of interest we can assume TE. That temperature is that of the particles, which follow a Maxwellian distribution with a single temperature, for all particle species. In other words, the temperature gradient scale must be small compared to the mean free path of the particles.

This probability density function gives the probability, per unit speed, of finding the particle with a speed near v. If particles follow a MB distribution, we can characterize them with a single temperature. When does this happen? When frequent collisions are able to thermalize the distribution. Particles at velocity v, move one "mean free path" λ in time t:

$$v = \frac{\lambda}{t} \tag{1}$$

The mean free path is

$$\lambda = \frac{1}{n\sigma}\,, \tag{2}$$

where n is the particle density and σ is the effective cross section (not necessarily the geometric cross section). Therefore,

$$t \simeq \frac{1}{n\sigma v}\,, \tag{3}$$

the particle timescale. This is a useful, although very approximate quantity! This sets the timescale over which a population of particles can thermalize.

The mean free path is related to the optical depth:

$$\tau_\nu = \int \kappa ds = \int n(s)\sigma ds \simeq \frac{s}{\lambda} \tag{4}$$

This is telling us something fundamental: when $\tau_\nu = 1$, the photons have traveled one mean free path. Because more photons will have traveled less than one mean free path than more, the mean distance is $< \lambda$.

2 Conservation of mass

We can invoke basic conservation laws, those of mass, momentum, angular momentum, and energy. Mass is the easy one. Assuming stars are spherically symmetric, the mass can be found via integration of the density:

$$m(r) = \int_0^r 4\pi r^2 \rho(r) dr\,, \tag{5}$$

or

$$dm = \rho 4\pi r^2 dr \tag{6}$$

This is our first equation of stellar structure!

3 The Energy Equation

Our book does a rather long derivation of the energy equation, but it seems unnecessary, so I'm just going to skip to the end result.

Stars can change energy (u) by doing work (W) or by absorbing heat (Q):

$$\delta(udm) = \delta Q + \delta W \tag{7}$$

and

$$\delta W = -P\delta dV = -P\delta\left(\frac{dV}{dm}dm\right) = -P\delta\left(\frac{1}{\rho}\right)dm \tag{8}$$

4 Hydrostatic Equilibium

Stars emit energy via fusion, which causes a pressure, or force, outward from their cores (where fusion takes place). Gravity tries to contract stars, and so provides a pressure or force inwards.

If the star is changing size, $\ddot{r} \neq 0$. We can write

$$\ddot{r}\Delta m = -\frac{Gm\Delta m}{r^2} + P(r)dS - P(r+dr)dS\,, \tag{9}$$

where the first term on the RHS is the gravitational force, the second and third terms are the pressure difference, and dS is the cross-sectional area. For small changes, $P(r+dr) = (\partial P/\partial r)dr$ so

$$\ddot{r} = -\frac{Gm}{r^2} - \frac{1}{\rho}\frac{\partial P}{\partial r} \tag{10}$$

or swapping dm for dr using conservation of mass:

$$\ddot{r} = -\frac{Gm}{r^2} - 4\pi r^2\frac{\partial P}{\partial m} \tag{11}$$

These forces must be balanced in most stars, or else the stars would change in size. The balance of these two forces is called "hydrostatic equilibrium," and it is one of the most important topics in understanding stars. If $\ddot{r} = 0$,

$$\frac{dP}{dr} = -G\frac{m\rho}{r^2} = -\rho g\,, \tag{12}$$

where dP/dr is the radial change in the pressure outward from the core, M_r is the mass **interior** to radius r, $\rho(r)$ is the mass density **at** radius r, and $g = GM_r/r^2$. This is the second equation of stellar structure.

Using mass conservation, we can also write

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}\,. \tag{13}$$

[What is the central pressure of the Sun?]

Equation 12 assumes that the star is stable, not contracting or expanding, which is the case for stars on the main sequence. If the internal pressure is too great, the condition of hydrostatic equilibrium fails and the star must expand or contract. We will return to this point when discussing stellar evolution. This is the first equation of stellar interiors.

This equation states that in order to balance gravity, there must be a pressure gradient! The minus sign shows that this gradient is such that pressure is highest at low r (in the star's core) and lowest at high r (at the star's surface).

5 The Virial Theorem

The Virial Theorem is fundamental. It relates the potential and kinetic energy in a gravitationally bound system. We can derive the Virial Theorem a bunch of ways, but let's start with the hydrostatic equation, $dP/dm = -Gm/(4\pi r^4)$. If we multiply this by a spherical volume $V = 4/3\pi r^3$ and integrate, we get

$$\int_0^{P(R)} V dP = -1/3\int_0^M \frac{GMdm}{r}\,, \tag{14}$$

where the LHS is related to the kinetic energy (U) and the RHS is 1/3 of the potential (Ω). We can integrate the LHS by parts

$$\int_0^{P(R)} V dP = PV\Big|_0^R - \int_0^{V(R)} PdV = -\int_0^{V(R)} PdV\,. \tag{15}$$

Therefore,

$$-3\int_0^{V(R)} PdV = \Omega\,. \tag{16}$$

Since $dV = dm/\rho$,

$$-3\int_0^M \frac{P}{\rho}dm = \Omega\,. \tag{17}$$

For an ideal gas, $u = 3/2kT/m_g = 3/2P/\rho$, so we are left with $2U - \Omega = 0$, which is the usual form of the Virial Theorem. We can also write that the total energy $E = U + \Omega$, or

$$E = 1/2\Omega = -U \tag{18}$$

6 The Gravitational Potential

We need a functional form for the potential. We know from before that

$$\Omega = \int_0^M \frac{GMdm}{r} . \tag{19}$$

If we take $m(r) = 4/3\pi r^3\bar{\rho}$, $dm = 4\pi r^2\bar{\rho}dr$ and then

$$\Omega = -\int_0^R G\frac{(4\pi)^2}{3}r^4\bar{\rho}^2dr = G\frac{(4\pi)^2}{3}\frac{1}{5}r^5\bar{\rho}^2\Big|_0^R = \frac{(4\pi)^2G}{3^2\times 2}\bar{\rho}^2R^6 . \tag{20}$$

If we substitute back in $\bar{\rho} = M/(4/3\pi r^3)$,

$$\Omega = \frac{3}{5}\frac{GM^2}{R} . \tag{21}$$

This is the potential for a constant density. In general, we can write

$$\Omega = \alpha\frac{GM^2}{R} . \tag{22}$$

As we can see, α depends on the density function. We'll always get a value near unity though.

7 Composition

Stars vary in metallicity, which we need some way to account for. Each element has A, the mass number (# of nucleons) and Z, the atomic number (# protons or electrons). For element i,

$$X = \frac{\rho_i}{\rho} , \tag{23}$$

the ratio of that elements mass density to the total mass density. Therefore

$$n_i = \frac{\rho_i}{A_i m_H} = \frac{\rho}{m_H}\frac{X_i}{A_i} , \tag{24}$$

and finally

$$X_i = n_i\frac{A_i}{\rho}m_H . \tag{25}$$

For compactness, let's define a variable to contain all the X values for all the elements:

$$\vec{X} = (X_1, ..., X_n) \tag{26}$$

We now have three equations that together describe stellar evolution:

$$\ddot{r} = -\frac{Gm}{r^2} - 4\pi r^2 \frac{\partial P}{\partial m} \tag{27}$$

$$\dot{U} + P\dot{\left(\frac{1}{\rho}\right)} = q - \frac{\partial F}{\partial m} \tag{28}$$

$$\dot{\vec{X}} = \vec{f}(\rho, T, \vec{X}) \tag{29}$$

We can add $dm = \rho r \pi r^2 dr$. To use these equations, we need functions for $P, U, F, q, \vec{f}$... which we often don't have. We can set up boundary conditions so that $P \to 0$ at $r = R$ and $F \to 0$ at $r = 0$, but that doesn't help much.

8 Timescales

For a long time, astronomers tried to figure out how stars made energy. Remember that we didn't even know about atoms until about a hundred years ago. Below we will describe some possible timescales.

We can characterize timescales as the relevant quantity divided by the change in that quantity, for example, the speed divided by the length, $\tau = \phi/\dot{\phi}$. If we have a 2m stretched out slinky and it is shortening at 0.5m/s, $\tau = 8\,\mathrm{s}$.

8.1 Dynamical

The dynamical timescale refers to the characteristic time required for a star to change its size. Thus, $\phi = R$. Our book determines $\dot{\phi}$ from the escape velocity

$$v_{\text{esc}} = \dot{\phi} = \sqrt{\frac{2GM}{R}} \tag{30}$$

Therefore,

$$\tau_{\text{dyn}} \approx \frac{R}{V_{\text{esc}}} = \left(\frac{R^3}{2GM}\right)^{1/2} \tag{31}$$

or, if $\bar{\rho} = M/(4/3\pi R^3$,

$$\tau_{\text{dyn}} \approx (G\bar{\rho}^{-1/2} \tag{32}$$

This is also known as the "free-fall time," the fastest that something can collapse if there are no other forces slowing it down. (The full derivation on your homework will give a factor of order unity out front.)

For the Sun, $t_{\text{dyn}} \approx 1000\,\mathrm{s}$ and in general

$$\tau_{\text{dyn}} \approx 1000\sqrt{\left(\frac{R}{R_\odot}\right)\left(\frac{M}{M_\odot}\right)} \tag{33}$$

As we can see, denser stars have shorter timescales.

This timescale is too short to provide energy to the star throughout the course of its life (it does, however provide energy once a star exhausts its nuclear fusion).

8.2 Thermal

This is also known as the Kelvin-Helmholtz timescale after the two physicists who derived it. In this case we are considering gravitational potential energy and luminosity (from an unspecified mechanism).

$$\phi = U \approx \frac{GM^2}{R} \tag{34}$$

and

$$\dot{\phi} = L \tag{35}$$

so

$$\tau_{\text{th}} \approx \frac{GM^2}{RL} \tag{36}$$

For the Sun, this works out to about 30 myr. Obviously, this cannot be the source of energy for stars on the main sequence either. In general,

$$\tau_{\text{th}} \approx 30 \text{ Myr} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{R}{R_\odot}\right) \left(\frac{L}{L_\odot}\right) \tag{37}$$

People thought this was how stars worked, but of course there are Earth rocks that are older, so something was amiss.

8.3 Nuclear

We know today that stars create energy via nuclear fusion.

The nuclear energy is $\phi = \epsilon Mc^2$ where $\epsilon = 0.007$ is the efficiency of converting hydrogen into helium via fusion. As before, $\dot{\phi} = L$. From before, we know that only 10% of the star has a high enough temperature to participate in fusion, so

$$\tau_{\text{nuc}} \approx 0.1\frac{\epsilon Mc^2}{L} = 10 \text{ Gyr} \left(\frac{M}{M_\odot}\right) \left(\frac{L}{L_\odot}\right) \tag{38}$$

So the numbers work out! This is of course what powers stars.

If we can assume hydrostatic equilibrium, our equations simplify greatly:

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} \tag{39}$$

$$\frac{\partial F}{\partial m} = q \tag{40}$$

$$\dot{\vec{X}} = \vec{f}(\rho, T, \vec{X}) \tag{41}$$