

# ASTR 367

## Stellar Spectra

Reading: (C+O) Chapters 8+9

Astronomy relies on the collection of light. Unlike most physical science disciplines, we cannot modify what we observe, only how we observe it.

We will focus here on “spectra,” which are measurements of intensity as a function of frequency or wavelength.

### Basic Definitions

Spectra can be classified as being composed of “continuum,” the smoothly varying curve, or “line,” which are sharper features seen in either absorption or emission. Continuous emission comes from processes where the energy levels (and hence photon energies) are not quantized. One example is a blackbody, which we talked about last time. A blackbody function is smooth, indicating that the energies are not quantized. In contrast, line emission relies on quantized energies. One example is electron orbital changes in an atom. The orbital energies are quantized, so the line appears at discrete frequencies.

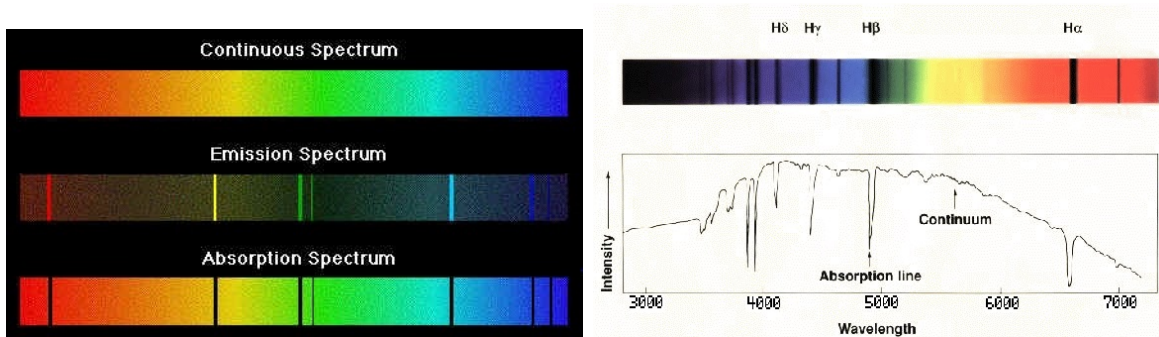


Figure 1: Examples of spectra.

### Bohr Model (mostly from Wikipedia)

But first, a bit more background....

In the early 20th century, experiments by Ernest Rutherford established that atoms consisted of a diffuse cloud of negatively charged electrons surrounding a small, dense, positively

charged nucleus. Given this experimental data, Rutherford naturally considered a planetary-model atom, the Rutherford model of 1911 - electrons orbiting a solar nucleus - however, said planetary-model atom has a technical difficulty. The laws of classical mechanics (i.e. the Larmor formula), predict that the electron will release electromagnetic radiation while orbiting a nucleus. Because the electron would lose energy, it would rapidly spiral inwards, collapsing into the nucleus on a timescale of around 16 picoseconds. This atom model is disastrous, because it predicts that all atoms are unstable.

To overcome this difficulty, Niels Bohr proposed, in 1913, what is now called the Bohr model of the atom. He put forward these three postulates that sum up most of the model:

- The electron is able to revolve in certain stable orbits around the nucleus without radiating any energy contrary to what classical electromagnetism suggests. These stable orbits are called stationary orbits and are attained at certain discrete distances from the nucleus. The electron cannot have any other orbit in between the discrete ones.
- The stationary orbits are attained at distances for which the angular momentum of the revolving electron is an integral multiple of the reduced Planck's constant:  $m_e v r = n \hbar$ , where  $n = 1, 2, 3, \dots$  is called the principal quantum number, and  $\hbar = h/2\pi$ . The lowest value of  $n$  is 1; this gives a smallest possible orbital radius of 0.0529 nm known as the Bohr radius. Once an electron is in this lowest orbit, it can get no closer to the proton.
- Electrons can only gain and lose energy by jumping from one allowed orbit to another, absorbing or emitting electromagnetic radiation with a frequency  $\nu$  determined by the energy difference of the levels according to the Planck relation:  $\Delta E = E_2 - E_1 = h\nu$  where  $h$  is Planck's constant.

Rydberg experimentally derived an expression for the wavenumber emitted by transitions of hydrogen atoms. This expression agreed with Bohr's theory, and can be generalized to all atoms as:

$$\frac{1}{\lambda} = R \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right). \quad (1)$$

This is known as the Rydberg formula, and the Rydberg constant  $R$  is

$$R = \frac{m_e e^4}{8\epsilon_0^2 h^2} \times \frac{1}{1 + m_e/M}, \quad (2)$$

where  $M$  is the total mass of the nucleus. The first term in this expression is defined as  $R_\infty$ . We can also define the Rydberg constant for hydrogen:  $R_H = R_\infty \frac{m_p}{m_e + m_p}$ .

We could rewrite the above expressions in terms of energy, since for photons  $E = hc/\lambda$ , so  $1/\lambda = E/hc$ .

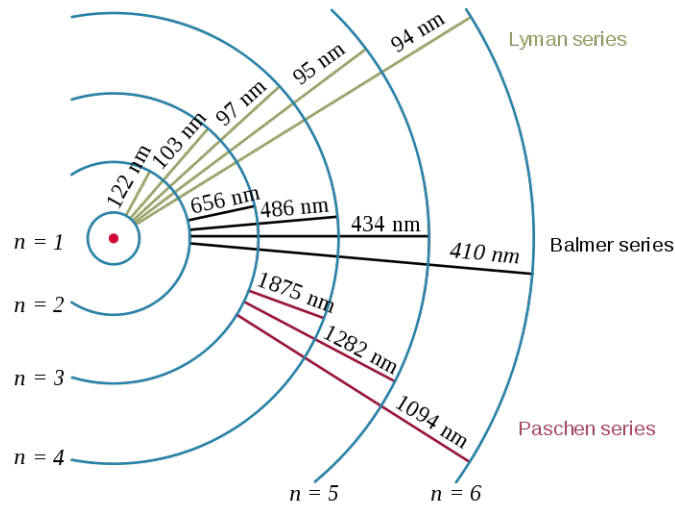


Figure 2: Electronic transitions of hydrogen.

## Kirchoff's Laws

We will mostly concern ourselves with line emission from electronic transitions here. In such cases, if an atom absorbs a photon with energy corresponding to the difference between two electronic orbitals, the electron will transition from the lower to the higher orbital. This is absorption, creating an absorption line; photons of that specific energy are removed from the path. (Aside: these same photons can be emitted when the electron transitions back to the lower level, but they do so isotropically, in  $4\pi$  steradians.) The opposite is emission, creating an emission line.

It is important to remember that absorption and emission are discrete properties, leading to discrete spectral lines.

The spectrum you observe depends on the density (technically the optical depth; see next section) of the object, and the viewing direction. Observing the same object from a different direction or direction will give you a different signal. **Kirchoff's Laws** tell us how to interpret the spectra we observe. There are multiple sets of Kirchoff's Laws, so it is safe to assume that Kirchoff was wicked smart and interesting at parties.

Kirchoff's three laws of spectra are:

- A dense object produces light with a continuous (blackbody) spectrum. Kirchoff also coined the term blackbody radiation because he was a show-off. You emit blackbody radiation, with a peak in the infrared. You should be familiar with Wien's law.
- A hot diffuse gas produces an emission line spectrum due to electronic transitions within the gas. Fluorescent lights are a good example.

- A hot dense object surrounded by a cool tenuous gas (i.e., cooler than the hot object) produces an absorption line spectrum. The absorption lines are at exactly the same wavelengths as the emission lines for a given element, and are also due to electronic transitions.

Stars show absorption line spectra in almost all cases. Why?

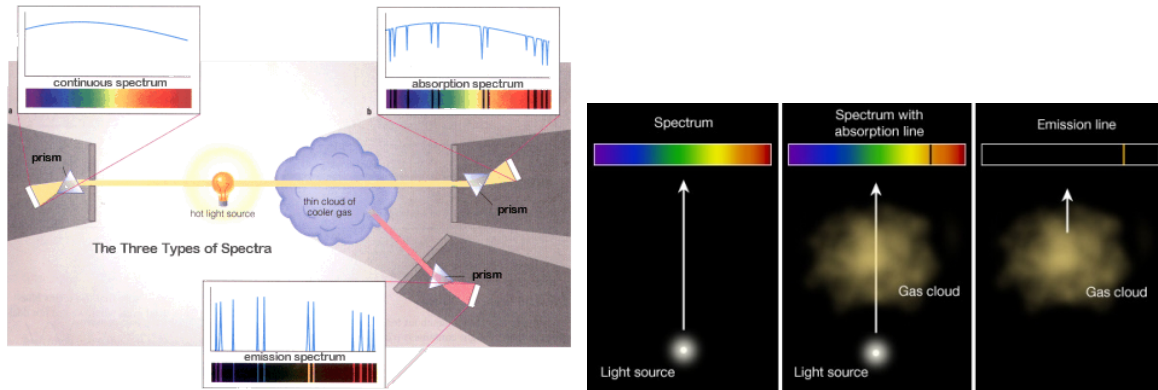


Figure 3: Illustrations of Kirchoff's Laws.

## Strength of Spectral Lines

The strength of spectral lines must depend on having elements present, and in the correct electronic configurations.

## Optical Depth

**Optical depth** ( $\tau$ ) goes from 0 (optically thin) to infinity (optically thick). A window, for example has an optical depth of approximately zero in the optical regime. A wall has a very high optical depth in the optical. *Optical depth is wavelength dependent, however.* Windows are opaque (optically thick) for UV radiation. Walls are transparent (optically thin) for X-rays. We usually think of an optical depth of approximately unity as being the transition between optically thick and optically thin. The concept of optical depth is a bit opaque (ha!), but is absolutely essential to understanding interstellar radiation.

If something is optically thick, you only see the top layer. On average, we see down to an optical depth of about 1. For optically thick material, this may not be very far into the material. For a wall, we reach an optical depth of unity within the thin layer of paint. If you want to estimate how much of something there is, you cannot easily do it if the object is optically thick. In this case, you don't know the total depth, in the same way that you don't know how thick the classroom walls are from observations within the classroom. If a material is optically thin, you get emission or absorption from the entire line of sight. If

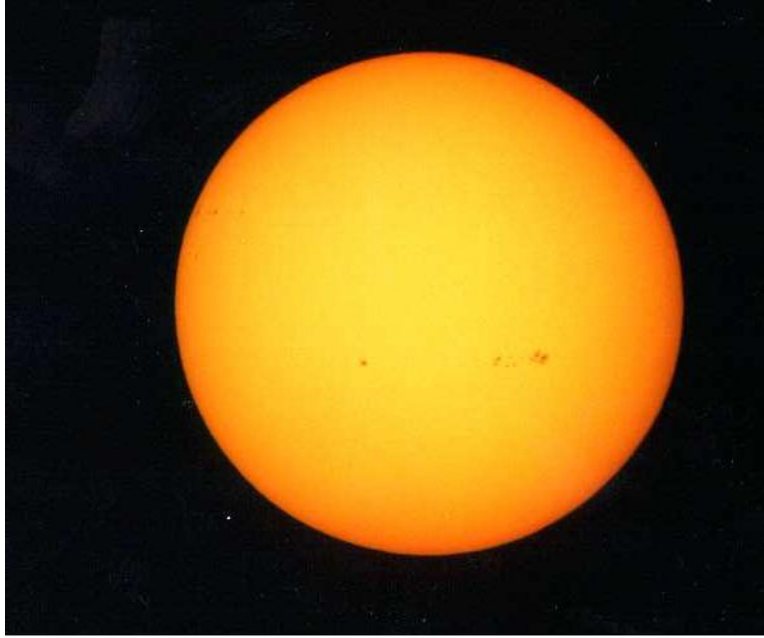


Figure 4: THE SUN! Notice the limb-darkening.

you know how the material emits or absorbs, you can estimate the amount of the emitting substance much more readily.

Optical depth is wavelength dependent ( $\tau_\nu$ ). For an emission line spectrum, the optical depth is zero where there is no line (and no continuum), and some positive value at the line (highest at line center). In other words, we can replace the intensity axis with  $\tau$ .

## Opacity

Optical depth is closely related to opacity,  $\kappa$ :

$$\tau_\nu = \int \kappa_\nu \rho ds, \quad (3)$$

where  $\rho$  is the mass density and the integration is over the path length. Opacity is intrinsic to the material, whereas optical depth is integrated along the path. We can see from the equation that opacity must have units of area/mass.

We can think of opacity as the analogue of emission. High opacity means that the element is able to absorb at that frequency. The element therefore must be in the correct electronic state (if applicable), and the correct ionization state (more on these later). The more of an element, the stronger the opacity. Therefore,  $\kappa_\nu = \kappa_\nu(\rho, T, X_i)$ .

Stellar opacity turns out to be incredibly important, as it is related to the luminosity. Opacity can remove photons from the medium (absorption) or simply redirect them and remove them from the line of sight (scattering).

There are five primary sources of opacity:

- Bound-Bound,  $\kappa_{\nu, \text{bb}}$ . The photon excites an electron and the electron then relaxes back to the ground state, releasing photons. These are sharp features, because only certain photons can cause the transitions. If there are multiple steps between the excited and ground states, multiple photons can be released, each of less energy than the original. The net result will be a reddening of the spectrum.
- Bound-free,  $\kappa_{\nu, \text{bf}}$ . These are photoionizing interactions. Any photon with energy greater than the ionization potential of the atom/molecule in question can photoionize it.
- Free-free absorption,  $\kappa_{\nu, \text{ff}}$ . Free electrons near ions can absorb photons over a continuous range of wavelengths.
- Electron (Thompson) scattering,  $\kappa_{\nu, \text{es}}$ . Electrons can also scatter, or change the direction of, photons. This results in continuum opacity.
- $\text{H}^-$  Continuum Opacity,  $\kappa_{\text{H}^-}$ . It may seem strange, but hydrogen can capture an extra electron, leading to something called  $\text{H}^-$ .  $\text{H}^-$  only forms in relatively low temperatures, say for F0 stars and later. The ionization potential of  $\text{H}^-$  is just 0.754 eV, which means that a large number of photons can ionize it. It is a significant source of opacity.

We can define a Rosseland Mean Opacity to attempt to determine an opacity averaged over all wavelengths that depends only on the temperature. Because the sources of opacity are complicated, this is not a trivial calculation! One result is in Figure 9.10 of your book.

We typically see no further into a star (or anything optically thick) than unity optical depth. A more careful treatment actually shows that the level within a stellar atmosphere from which most of the photons of wavelength  $\lambda$  escape is at optical depth  $\tau_\lambda \simeq 2/3$ . Indeed, the condition  $\tau_\lambda \simeq 2/3$  defines the stellar photosphere – the layer of a star's atmosphere from which the light we see originates.

There are two consequences of this realization. First, the condition applies to all viewing angles; therefore, the distance  $ds$  corresponding to the condition  $\tau_\lambda = 2/3$  will probe further into the star's interior at the center of a stellar disk than at its edges. Second, recalling the definition of optical depth, it is obvious that if the opacity  $\kappa_\lambda$  increases at some wavelength, then  $ds$  must be smaller to satisfy the condition  $\tau_\lambda = 2/3$ . Thus, we see further into a star in its continuum light than at the wavelengths of discrete absorption lines. These two effects explain a phenomenon known as limb darkening, first recognised in the Sun, whereby the light emitted in successive annuli from the centre decreases in intensity and becomes progressively redder. Sightlines near the limb do not penetrate as deeply into the Sun's atmosphere by the time  $\tau_\lambda = 2/3$  is reached; since the Sun's temperature decreases outwards from the center, such sightlines see light from cooler regions of the Sun's atmosphere.

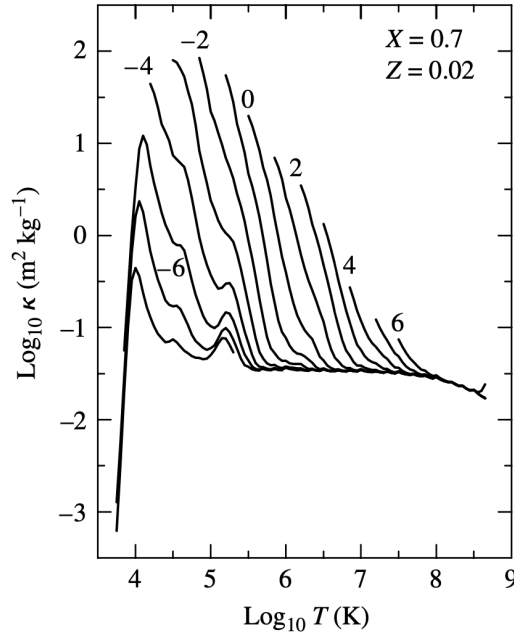


Figure 5: Computation of the Rosseland mean opacity. The curves are the logarithm of the density.

## Maxwell-Boltzmann Speed Distribution

What is a maxwellian distribution?

$$f(v) = \sqrt{\left(\frac{m}{2\pi kT}\right)^3} 4\pi v^2 e^{-\frac{mv^2}{2kT}}, \quad (4)$$

where  $m$  is the particle mass and  $kT$  is the product of Boltzmann's constant and thermodynamic temperature. This probability density function gives the probability, per unit speed, of finding the particle with a speed near  $v$ . The most probable speed (peak of  $f(v)$ ) is:

$$v_p = \sqrt{\frac{2kT}{m}}, \quad (5)$$

and the mean speed is

$$\langle v \rangle = \int_0^\infty v f(v) dv = \sqrt{\frac{8kT}{\pi m}} = \frac{2}{\sqrt{\pi}} v_p. \quad (6)$$

(from Wikipedia) The Maxwell-Boltzmann distribution applies to the classical ideal gas, which is an idealization of real gases. In real gases, there are various effects (e.g., van der Waals interactions, relativistic speed limits, and quantum exchange interactions) that make their speed distribution sometimes very different from the Maxwell-Boltzmann form. That said, rarefied gases at ordinary temperatures behave very nearly like an ideal gas and the Maxwell speed distribution is an excellent approximation for such gases.

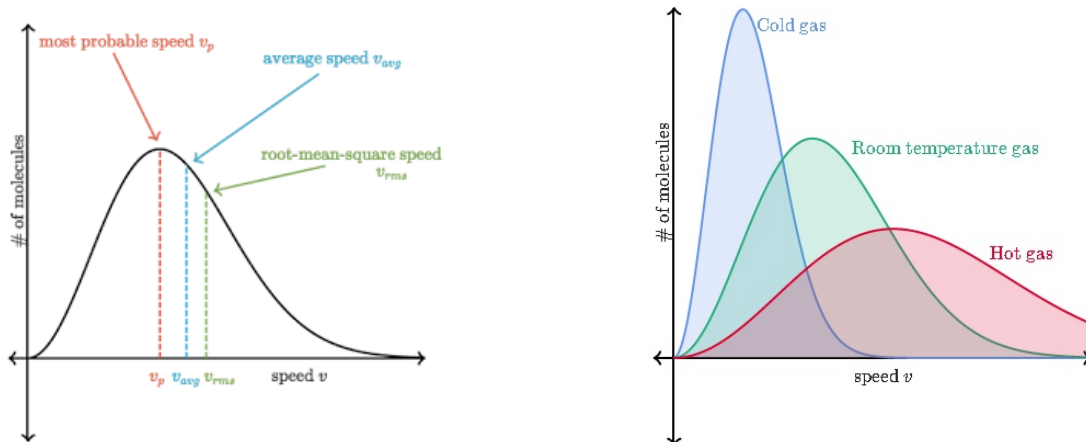


Figure 6: MB distribution illustrations.

This is in 3D! We actually measure only 1D. Keep this in mind for later.

What is the mean free path?

$$mfp = 1/n\sigma, \quad (7)$$

where  $n$  is the particle density and  $\sigma$  is the particle cross section. Another way of saying this is that collisions dominate in LTE over other (radiative) processes. Obviously, high density environments are more likely to be in LTE.

The mean time between collisions is  $mfp/v$ , where  $v$  can be from the M-B distribution.

So let's return to our Bohr model. The Bohr model says that photons are emitted or absorbed when electrons change energy levels. More energetic photons are created when those energy level changes are larger. How can we populate the upper levels in an atom? By collisions with fast moving particles! So as the temperature increases, different emission and absorption lines will dominate. We can tell from a star's spectrum what its temperature is!

Spectral Type	Surface Temperature	Distinguishing Features
O	> 25,000K	H; HeI; HeII
B	10,000-25,000K	H; HeI; HeII absent
A	7,500-10,000K	H; CaII; HeI and HeII absent
F	6,000-7,500K	H; metals (CaII, Fe, etc)
G	5,000-6,000K	H; metals; some molecular species
K	3,500-5,000K	metals; some molecular species
M	< 3,500K	metals; molecular species (e.g., TiO)



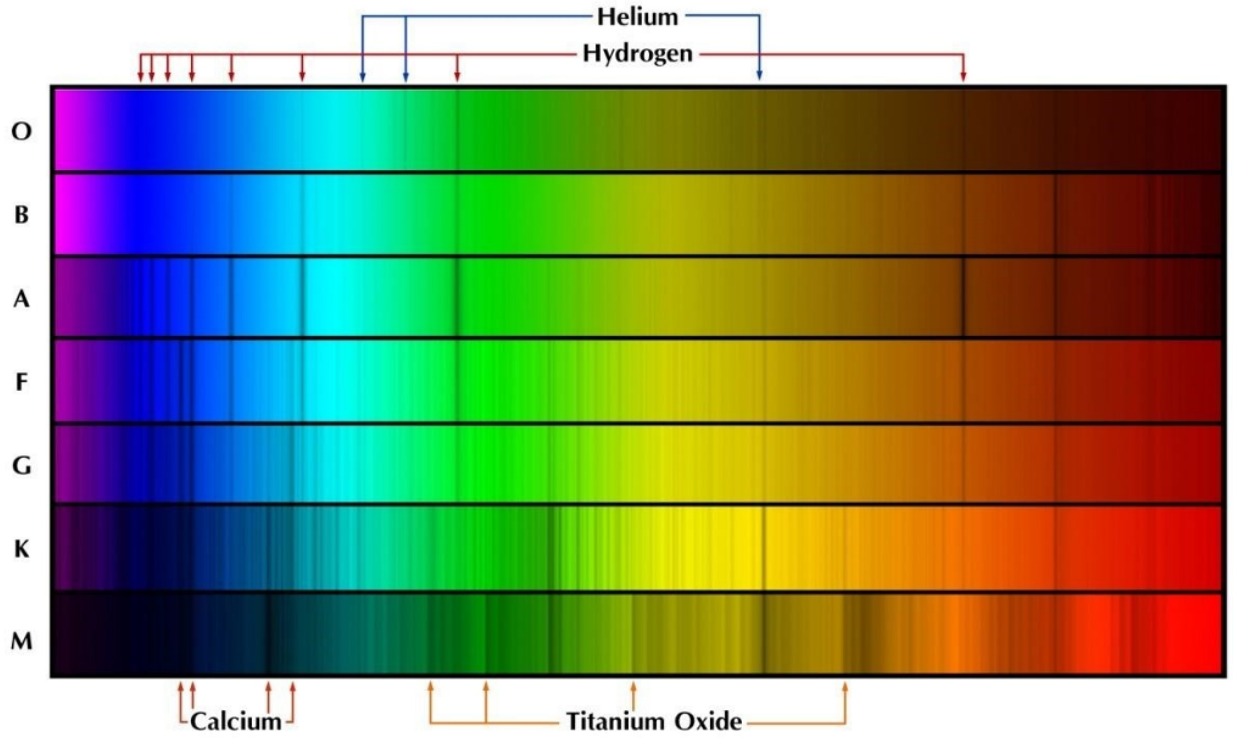


Figure 7: Stellar spectra.

## Boltzmann Equation

The single most important equation in stat. mech. for us is the Boltzmann Equation. We can define the relative density within state  $i$  as

$$n_i = g_i e^{-E_i/kT_{\text{ex}}}, \quad (8)$$

where  $n_i$  is the density in state  $i$ ,  $g_i$  is the degeneracy of state  $i$ ,  $E_i$  is the energy associated with that state and  $T_{\text{ex}}$  is the “excitation temperature.” If we want to know the relative density between two states,

$$\frac{n_i}{n_j} = \frac{g_i e^{-E_i/kT_{\text{ex}}}}{g_j e^{-E_j/kT_{\text{ex}}}} = \frac{g_i}{g_j} e^{-E_{ij}/kT_{\text{ex}}}, \quad (9)$$

where  $E_{ij} = E_i - E_j$  is the energy difference between the two states.

Let’s take hydrogen as an example, levels  $n = 1$  and  $n = 2$ . For hydrogen,  $E_n = 13.6 \text{ eV}/n^2$ . For  $n = 1$ ,  $E_1 = 13.6 \text{ eV}$ . For  $n = 2$ ,  $E_2 = 3.4 \text{ eV}$ . Therefore,  $E_{ij} = 13.6 \text{ eV} - 3.4 \text{ eV} = 10.2 \text{ eV}$ . Degeneracies  $g_n = 2n^2$  so  $g_1 = 2$  and  $g_2 = 8$ .

While this equation gives the relative densities between states, we are frequently interested in the fractional density of a given state compared to all states. In such cases, we need to

use the “partition function,”

$$Z(T_{\text{ex}}) = \sum_j^{\infty} g_j e^{-E_j/kT_{\text{ex}}} \quad (10)$$

So that

$$\frac{n_i}{n_T} = \frac{g_i e^{-E_i/kT_{\text{ex}}}}{Z(T_{\text{ex}})}, \quad (11)$$

where  $n_T$  is the total population.

The excitation temperature is not a physical temperature! It is instead the temperature at which the Boltzmann equation is satisfied. When is  $T_{\text{ex}} = T_k$  the kinetic temperature? When collisions are frequent! Assume we have two competing processes: collisions and radiation, and that the kinetic (collision) temperature  $T_k$  and radiation temperature ( $T_R$ ) are different. If the timescale for collisions is smaller than the timescale for photon-particle interactions,  $T_{\text{ex}} \simeq T_k$ .

For another example, assume that the population levels are inverted such that the upper level is overpopulated relative to the lower level, then  $T_{\text{ex}}$  is negative. This is allowed because  $T_{\text{ex}}$  is not a real temperature. These population inversions can result in masing emission.

It is also worth noting that the excitation temperature only corresponds to the transition between the upper and lower levels. Each transition can have a different excitation temperature.

Radiation temperature,  $T_R$ , is the the equivalent temperature blackbody that would emit the same intensity at the frequency of interest. This is also sometimes called the background temperature,  $T_{\text{BG}}$  or  $T_0$ . In the limit of low frequencies where the RJ limit applies, the brightness and radiation temperatures are the same.

IF  $T_k = T_R = T_{\text{ex}} = T$ , the system is in *thermodynamic equilibrium* (TE). This happens when the particle energy distribution follow the Boltzmann equation (and ionization states follow Saha, see below), the particle velocity distributions follow MB, and the radiation field is a Planck function at temperature  $T_R$ .

If  $T_k = T_{\text{ex}} \neq T_R$ , the system is in *local thermodynamic equilibrium*, LTE. LTE is much easier to attain and is commonly assumed. This is often good enough when collisions dominate over radiative processes. “Local” here refers to  $\sim$  one mean free path.

In LTE, the changes in temperature must vary slowly, so that at each point in the object of interest we can assume TE. That temperature is that of the particles, which follow a Maxwellian distribution with a single temperature, for all particle species. In other words, the temperature gradient scale must be small compared to the mean free path of the particles.

## Saha Equation

The Saha ionization equation relates the ionization state of an element to the temperature and pressure.

For a gas composed of a single atomic species in LTE, only concerning two states (excited and not as excited) the Saha equation is written:

$$\frac{n_{i+1}n_e}{n_i} \simeq 2 \left( \frac{2\pi m_e kT}{h^2} \right)^{3/2} \frac{g_{i+1}}{g_i} \exp \left[ -\frac{\Phi_r}{kT} \right], \quad (12)$$

where  $n_i$  is the density of atoms in the  $i$ -th state of ionization, that is with  $i$  electrons removed.

$g_i$  is the degeneracy of states for the  $i$ -ions

$\Phi_r$  is the energy required to remove  $i$  electrons from a neutral atom, creating an  $i$ -level ion (the “ionization potential”).

$n_e$  is the electron density

$m_e$  is the mass of an electron

$T$  is the temperature of the gas

$k_B$  is the Boltzmann constant

$h$  is Planck’s constant .

Hydrogen is particularly simple. The degeneracy for the ground state of hydrogen is 4 (proton spin up, electron up; p up e down; p down e up; p down e down). The degeneracy for the ionized state is 2. We therefore have:

$$\frac{n_{H^+}n_e}{n_H} \simeq \left( \frac{2\pi m_e kT}{h^2} \right)^{3/2} \exp \left[ -\frac{13.6 \text{ eV}}{k_B T} \right], \quad (13)$$

## The Equation of Radiative Transfer

Now that we have some background, let’s consider radiation with intensity  $I_\nu$  entering a slab of material that has a thickness  $ds$ . On the other side of the slab the intensity is  $I_\nu + dI_\nu$ , where  $dI_\nu$  could be negative. We can describe the radiation thusly:

$$dI_\nu = -I_\nu \kappa_\nu \rho ds + j_\nu ds. \quad (14)$$

The term  $-I_\nu \kappa_\nu \rho ds$  is the net change in  $I_\nu$  due to absorption and stimulated emission and  $j_\nu ds$  is the change in  $I_\nu$  due to spontaneous emission by the slab material.  $\kappa_\nu$  is the opacity at frequency  $\nu$ . The attenuation coefficient is normally positive, but can be negative for masers. It has dimensions of 1/length.  $j_\nu$  is the emissivity at frequency  $\nu$ . It also has dimensions of power per unit volume per unit frequency per unit solid angle.

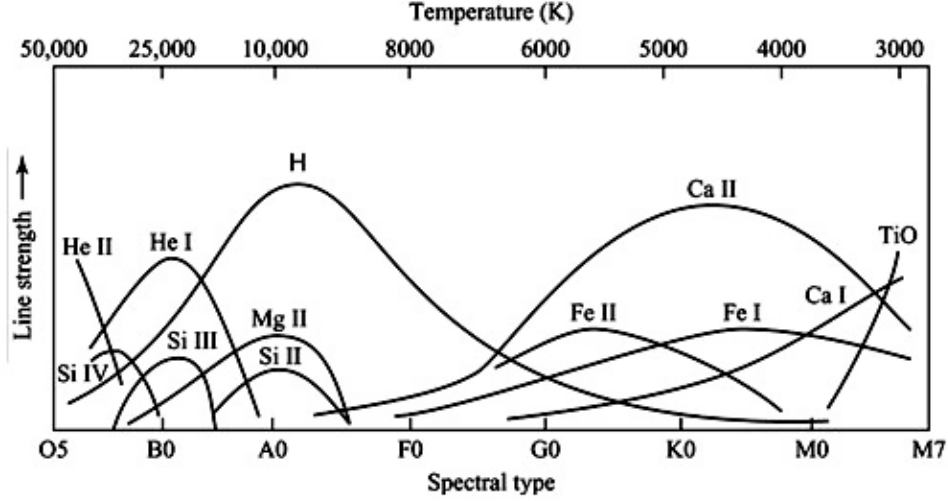


Figure 8: Spectral types and common spectral lines.

The optical depth is intimately related to the opacity. This makes sense. Something has to be causing the optical depth!

$$d\tau_\nu = \kappa_\nu \rho ds \quad (15)$$

or

$$\tau_\nu = \int_0^s \kappa_\nu \rho ds = \kappa_\nu \rho s \quad (16)$$

so therefore

$$dI_\nu = -I_\nu d\tau_\nu + S_\nu d\tau_\nu. \quad (17)$$

We introduced a new term,  $S_\nu$ , which is the source function,

$$S_\nu = \frac{j_\nu}{\kappa_\nu \rho}. \quad (18)$$

The source function describes the change in emission along the line of sight. In LTE,  $S_\nu(T) = B_\nu(T) = j_\nu / \kappa_\nu \rho$ . This is another powerful thing about LTE: it directly relates measurements to the emission and absorption properties of the material.

We can integrate the transfer function by multiplying by  $e^\tau$ , because it works.

$$e^{\tau_\nu} (dI_\nu + I_\nu d\tau_\nu) = e^{\tau_\nu} S_\nu d\tau_\nu \quad (19)$$

$$d(e^{\tau_\nu} I_\nu) = e^{\tau_\nu} S_\nu d\tau_\nu \quad (20)$$

If we define  $\tau_\nu = 0$  at  $I_\nu(0)$ ,

$$e^{\tau_\nu} I_\nu - I_\nu(0) = \int_0^{\tau_\nu} e^{\tau'} S_\nu d\tau' \quad (21)$$

multiply by  $e^{-\tau_\nu}$

$$I_\nu(\tau_\nu) = I_\nu(0) e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{\tau_\nu - \tau'} S_\nu d\tau'. \quad (22)$$

The intensity  $I_\nu$  at optical depth  $\tau_\nu$  is the initial (background) intensity  $I_\nu(0)$  attenuated by a factor  $e^{-\tau_\nu}$ , plus the emission  $S_\nu d\tau'$  integrated over the path, itself attenuated by the factor  $e^{\tau_\nu - \tau'_\nu}$ .

This is the most general form. If LTE holds,  $S_\nu = B_\nu(T)$ . Furthermore, if  $T$  is constant throughout the slab, we can simplify Equation 22 considerably:

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + B_\nu(T)(1 - e^{-\tau}). \quad (23)$$

This is the usual form used, but assumes LTE and a constant temperature throughout the slab. These are not exactly the same thing since the former depends on the mean free path (the density), and the latter depends on the slab thickness.

It is worth examining this equation a bit more in limiting cases:

(1) If the slab optical depth is zero, we get  $I_\nu = I_\nu(0)$ , simply the background intensity back. If there is no optical depth, we get neither emission nor absorption from the slab (like a window!). This illustrates how emission and absorption are intimately related.

(2) If the slab optical depth is infinite, we get  $I_\nu = 0 + B_\nu(T)$ . In this case, there is no radiation from the background. It is all absorbed by the back side of the slab. Optical depth cuts both ways: if it is optically thick for us as observers, it is optically thick for the background radiation as well.

Interesting features:

(1) As  $\tau \rightarrow 0$ ,  $e^{-\tau} \rightarrow 1 - \tau$ . Thus,  $I_\nu \rightarrow I_\nu(0)(\tau - 1) + B_\nu(T)\tau$  and  $\Delta I_\nu \rightarrow \tau_\nu(B_\nu(T) - I_\nu(0))$ . In the absence of background emission, we just measure the excitation temperature multiplied by the optical depth.

(2) As  $\tau \rightarrow \infty$ ,  $I_\nu \rightarrow B_\nu(T)$  and  $\Delta I_\nu \rightarrow B_\nu(T) - I_\nu(0)$ . For optically thick materials, the measured antenna temperature is simply the Planck function! Thus, for optically thick materials, we can readily estimate their temperatures.

(3) If the Planck function of the material is greater than the background temperature, the line is seen in emission.

(4) If the excitation temperature is less than the background temperature, the line is seen in absorption.

## Line Broadening

Emission and absorption lines are not delta functions, they are broadened by some mechanisms. What can broaden spectral lines?

1) Thermal doppler motion due to gas particles at a given temperature, ‘‘Doppler broadening.’’ Not all particle speeds will be the same of course. As the temperature increases, the range of speeds does too.

2) ‘‘Turbulent broadening’’ due to the fact that there are bulk motions within any gas.

- 3) Natural broadening due to the fact that the energy “level” is not a single value, and
- 4) Pressure or collisional broadening, which changes the energy levels.

Each of these mechanisms has an associated line shape. Doppler broadening is Gaussian, and turbulent is usually assumed to be Gaussian as well. Natural and Collisional broadening are “Lorentzian,” which is like a Gaussian but with much larger “wings.” All four processes operate at the same time, resulting in a “Voigt” profile with a Gaussian core and Lorentzian wings. Since the Lorentzian wings are at low intensity, usually a Gaussian is observed.

Gaussians are magical functions. A normalized Gaussian takes the form of

$$\phi(\nu) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{\nu-\nu_0}{2\sigma}\right)^2}, \quad (24)$$

where  $\nu_0$  is usually the line center and  $\sigma$  is the one-dimensional velocity dispersion. You can see that the line will be of maximum intensity when  $\nu = \nu_0$  at line center, then  $\phi(\nu_0) = \frac{1}{\sigma\sqrt{\pi}}$ .

We can also define the full-width at half-maximum (FWHM) as

$$FWHM = \sqrt{8 \ln 2} \sigma = 2.355 \sigma. \quad (25)$$

Gaussians have the amazing property that the area under the curve is approximately the FWHM times the peak. They are also their own Fourier transform pair, and a Gaussian convolved with a Gaussian leads to another Gaussian. This last point is very important in astronomy. Frequently, the response of your instrument can be assumed to be Gaussian, and the source or spectral line Gaussian as well. Therefore, you will observe a Gaussian.

Why do we get Gaussians? We had the MB velocity distribution before:

$$f(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2 e^{-mv^2/2kT}. \quad (26)$$

What you have probably never seen is that in 1D the  $4\pi v^2$  term goes away. This term arises in 3D due to the density of velocity states available (see Hyperphysics site). This is a Gaussian!

We can define:

$$\sigma_v = \left(\frac{kT}{m}\right)^{1/2} = 9.12 \left(\frac{T_4}{m/\text{amu}}\right)^{1/2} \text{ km/s}, \quad (27)$$

where  $T_4$  is the temperature in units of  $10^4$  K. The FWHM is  $2.355\sigma$ , or

$$FWHM = 2.355\sigma = 21.47 \left(\frac{T_4}{m/\text{amu}}\right)^{1/2} \text{ km/s}. \quad (28)$$

We can convert between velocity and frequency/wavelength using the Doppler formula

$$\frac{\Delta\lambda}{\lambda} = \frac{\Delta v}{c}, \quad (29)$$

were  $\Delta\nu$  is the broadening in frequency,  $\nu$  is the central frequency of the line, and  $v$  is the broadening in velocity units. This leads to:

$$f(v) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_v} e^{-v^2/2\sigma_v^2}. \quad (30)$$

You can see how the velocity dispersion  $\sigma_v$  goes into the Gaussian. The thermal line width increases with increasing temperature and decreases with increasing mass.

We can assume that turbulence adds another Gaussian term, and we can add the thermal and turbulent components in quadrature, which leads to

$$\sigma = \left( \frac{kT}{m} + v_{\text{turb}}^2 \right)^{1/2}, \quad (31)$$

We often don't know  $v_{\text{turb}}$ , but given a spectral line we can determine it from the linewidth if we can estimate the temperature.

The natural width arises due to the uncertainty principle:  $\Delta E \Delta t \sim \hbar$ . Here,  $\Delta t = A_{\text{ul}}^{-1}$ , so short-lived states have large uncertainties in energy. Because  $A_{\text{ul}} \propto \nu^3$ ,  $\Delta E \propto \nu^3$ . Natural broadening is important at high frequencies. It gives rise to a Lorentzian profile function

$$\phi(\nu) = \frac{\gamma}{4\pi^2} \frac{1}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2}, \quad (32)$$

where  $\gamma$  is a constant for each species related to the spontaneous decay rates

$$\gamma = \Sigma A_{\text{ul}}. \quad (33)$$

Like natural broadening, collisional broadening distorts the energy levels, leading to an additional Lorentzian term that can be combined with natural broadening:

$$\phi(\nu) = \frac{\Gamma}{4\pi^2} \frac{1}{(\nu - \nu_0)^2 + (\Gamma/4\pi)^2}, \quad (34)$$

where  $\Gamma = \gamma + 3\nu_{\text{col}}$ , and  $\nu_{\text{col}}$  is the collision frequency,  $\nu_{\text{col}} = n\sigma v$ .

In the radio, Doppler and turbulent broadening are typically the largest. At high frequencies (X-ray), natural broadening can make a large contribution.

## Characterizing Spectral Lines

It is often preferable to fit a line profile to a spectral line in order to characterize its emission. An alternative method, called the "equivalent width," is insensitive to the exact profile. The equivalent width is the width, in wavelength or frequency units of a rectangular area equal to that of the spectral line:

$$W_\lambda = \int (1 - F_\lambda/F_0) d\lambda \quad (35)$$

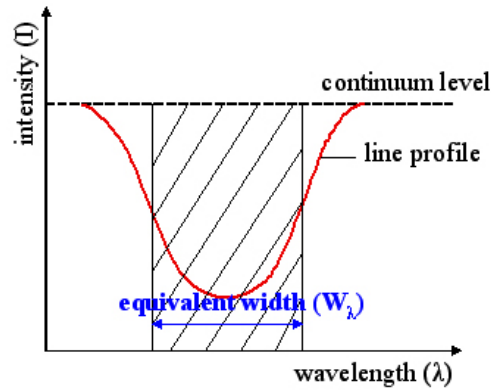


Figure 9: Equivalent width.

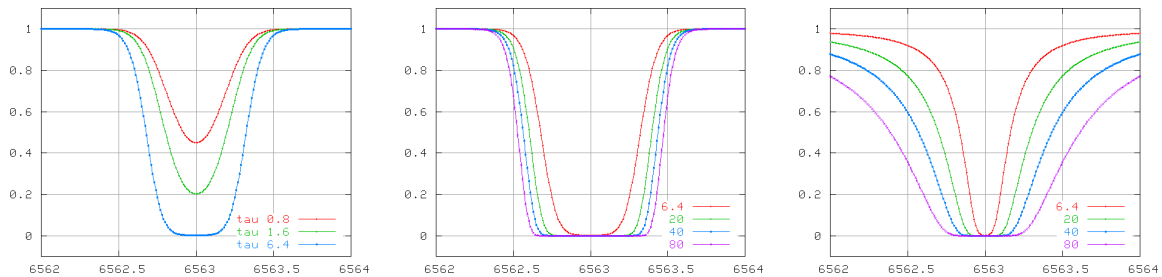


Figure 10: Line saturation for various optical depths (<http://spiff.rit.edu/classes/phys440/lectures/curve/curve.html>).

The intensity of a spectral line changes with the optical depth at line center. This is usually parameterized as the change in the optical depth, or the change in “column density,” the integral of the number density over the pathlength.

Let’s assume we have an absorption line and no source function:

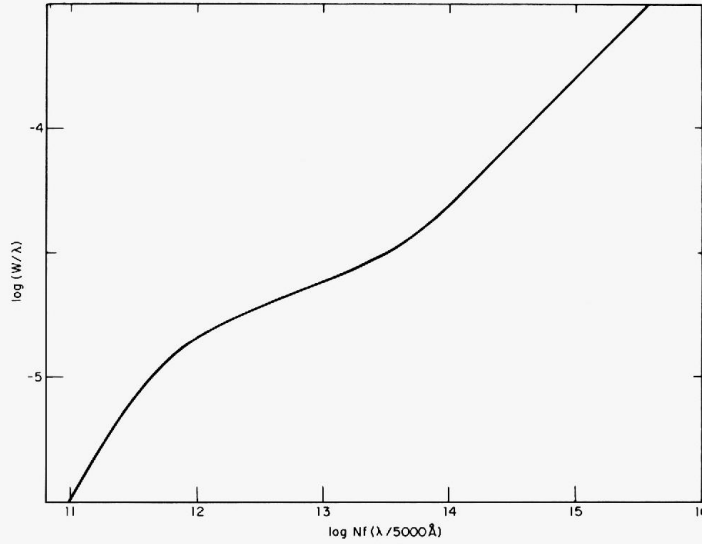
$$\frac{I_\nu}{I_{\nu,0}} = e^{-\tau_\nu} \quad (36)$$

Initially, the more absorbing atoms there are, the stronger the absorption line. If we keep adding absorbers to the path, however, eventually the line will saturate. A saturated line means that at line center (the most probable speed in a MB distribution) no more intensity can be added. After it saturates, the equivalent width grows slowly, because there are few atoms with the requisite speeds, until the growth of the line wings becomes important.

The column density is a fundamental quantity in astronomy, since it is directly related to what we measure. Since  $\kappa_\nu \propto n$  and  $\tau_\nu = \int \kappa_\nu \rho ds$ ,  $\tau \propto \int n ds = N$ . This is the number of particles along a  $1 \text{ cm}^2$  cylinder path.

This leads to three important regimes:





**Figure 9.22** A general curve of growth for the Sun. (Figure from Aller, *Atoms, Stars, and Nebulae*, Revised Edition, Harvard University Press, Cambridge, MA, 1971.)

Figure 11: The curve of growth. The x-axis is parameterized in terms of the oscillator strength, which is a parameter unique for each transition.

- 1) The “Linear” regime ( $\tau \lesssim 5$ ) where the equivalent width  $W$  is proportional to the column density,  $W \propto N$ .
- 2) The “saturated” regime where  $W \propto \sqrt{\ln N}$ , and
- 3) The “damping” regime where collisional broadening takes over,  $W \propto \sqrt{N}$ .