Compact Objects, Prialnik Ch 9 for WDs

White Dwarfs, Neutron stars and Black Holes

Compact objects are the end products of stellar evolution. The primary factor determining whether a star ends up as a white dwarf, neutron star, or a black hole is the star's mass.

Compact objects differ from normal stars in that they are not supported against collapse by fusion. In a white dwarf, the gravity is balanced by the electron degeneracy pressure. However, once the density of matter approaches nuclear density, the degeneracy pressure of neutrons becomes important (at such high density, inverse beta decay converts protons into neutrons). A neutron star is thus supported by the degeneracy pressure of neutrons. Black holes, on the other hand, are completely collapsed stars, i.e., stars that could not find any means to hold back the inward pull of gravity and therefore collapsed to singularities.

White Dwarfs

For low-mass stars ($\lesssim 8\ M_\odot$), the core is left behind when the stars go planetary nebula. These cores are then white dwarfs: very small, very hot, very low luminosity objects. WDs result from AGB stars.

Usually, white dwarfs are composed of carbon and oxygen and the progenitor mass is less than 8 $M_\odot$. Some flavors of white dwarf do have different compositions though.

If the mass of the progenitor is between 8 and 10.5 solar masses, the core temperature will be sufficient to fuse carbon but not neon, in which case an oxygen/neon/magnesium white dwarf may form. Although helium in most white dwarfs could be fused, this isn't always true for low mass stars. Stars of very low mass may accrete He from a binary companion, and so may have He in their outer layers.

Exercise

Let's compute the basic properties of white dwarfs! Take a 20,000 K White Dwarf on the 0.5 $M_\odot$ line and compute its
- Radius (from the luminosity and temperature. Earth's radius is $\sim$6000 km)
- Density assuming its mass is 0.5 $M_\odot$ (density of the Earth is $\sim 5\,\mathrm{g\,cm^{-3}}$ and water is $1\,\mathrm{g\,cm^{-3}}$)
- Central pressure (from hydrostatic equilibrium assuming constant density)

We can also use the expression derived in-class for constant density:

$$P_c \approx 2/3\pi G\rho^2 R^2 \tag{1}$$

For the mass of the Sun and radius of the Earth, this works out to $P_c \approx 4 \times 10^{22}\,\mathrm{N\,m^{-2}}$, which is $\sim 1.5 \times 10^6$ times that of the Sun.

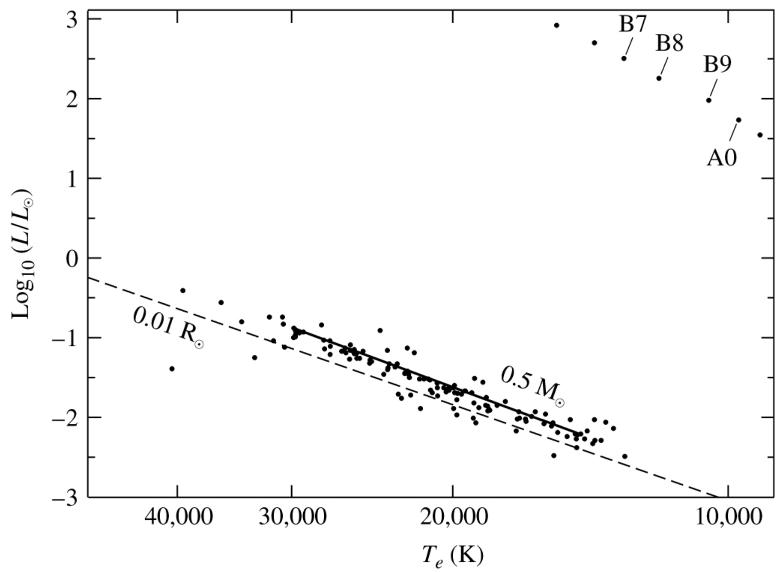


Figure 1: A portion of the H-R diagram showing white dwarfs. Notice that the 0.5 $M_\odot$ line is sloping toward the 0.01 $R_\odot$ line. What does that imply?

WD Temperatures

We can make similar arguments for non-convective stars:

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T^3}\frac{L_r}{4\pi r^2} \tag{2}$$

It is not radiation that carries energy to the surface, but rather electron conduction. We can still approximate things with this relation though. The temperatures go from the surface temp to the central temperature. The radius goes from the radius of the white dwarf to zero, so

$$\frac{T_{wd} - T_c}{R - 0} = -\frac{3}{4ac}\frac{\bar{\kappa}\rho}{T_c^3}\frac{L_r}{4\pi R^2} \tag{3}$$

We can get the surface temperature from observations, and R from the above calculations. If, however, we assume the surfact temperature is 0 K and that $\kappa = 0.02\,\mathrm{m}^2\ \mathrm{kg}^{-1}$, and we get $T_c \approx 10^7 - 10^8$ K. This temperature is plenty high for hydrogen fusion. Since white dwarfs do not have fusion, we know that they must be largely devoid of hydrogen. What little hydrogen they have is on the surface; the more massive elements are drawn toward white dwarf cores.

WD Luminosities

The Big Orange Book derives an expression for the WD luminosity, but it's kind of a long derivation and not terribly informative. The result is interesting though:

$$L/L_\odot = CT_c^{7/2}\,, \tag{4}$$

where C is a constant that is

$$C = 6.65 \times 10^{-3}\left(\frac{M}{M_\odot}\right)\frac{\mu}{Z(1+X)} \tag{5}$$

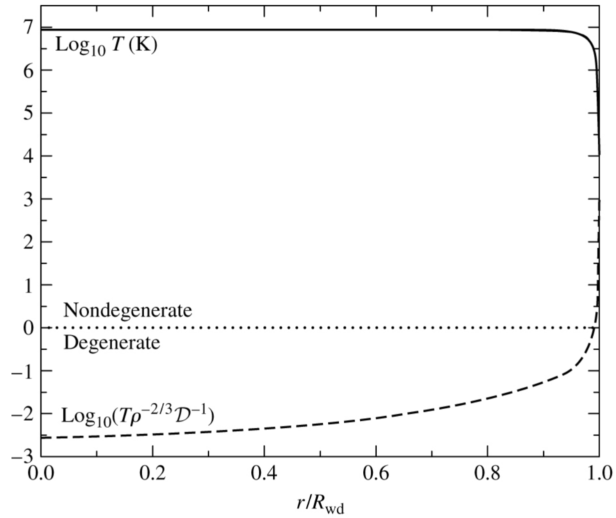


Figure 2: The temperature of WDs is isothermal out to the edge.

Or, following our book:

$$\frac{L/\,L_\odot}{M/\,M_\odot} \approx 6.8 \times 10^{-3} \left(\frac{T_C}{10^7\ \mathrm{K}}\right)^{7/2} \tag{6}$$

A white dwarf's faint luminosity comes from the emission of stored thermal energy; no fusion takes place in a white dwarf and there is no further contraction to release gravitational potential. Because white dwarfs are the end point of all low-mass stars, which are very numerous (over 97% of the other stars in the Milky Way), there should be lots of white dwarfs in the Universe. Because they are faint, however, they are difficult to find.

Types of White Dwarfs

Stamp collecting!
White Dwarfs can be classified based on their spectra. DA white dwarfs have hydrogen absorption lines in their spectra. These lines are extremely pressure broadened due to the high pressures in the white dwarf surfaces. White dwarfs are mostly carbon and oxygen, but some do have trace amounts of hydrogen. 2/3 of all WDs.
DB white dwarfs have helium absorption lines ,but lack hydrogen. 8% of all WDs.
DC white dwarfs have no lines, they are black bodies. 14% of WDs.

White Dwarf Cooling

Energy in white dwarfs does not escape most efficiently from photons. In fact, it is electron conduction that provides the dominant energy transportation method. In a WD, electrons can travel large distances before interacting with another nucleus. As a result, WDs are basically isothermal. The only place that is not isothermal is the outer shell of material.

So how does a WD cool? Well, the WD's energy is thermal, and each nucleus has $3/2kT$ of energy. Therefore, the thermal energy of a WD is

$$U = \frac{M}{Am_H}\frac{3}{2}kT_c\,, \tag{7}$$

where Am_H is the mass of one nucleus. The characteristic timescale (not the cooling time) is

$$\tau = \frac{U}{L} = \frac{3}{2}\frac{Mk}{Am_h C T_c^{5/2}} \tag{8}$$

The time to cool is given by

$$-\frac{dU}{dt} = L \tag{9}$$

or

$$-\frac{d}{dt}\left(\frac{M}{Am_H}\frac{3}{2}kT_c\right) = CT_c^{-7/2}\,, \tag{10}$$

which can be integrated to find

$$T_c(t) = T_0\left(1+\frac{5}{2}\frac{t}{\tau_0}\right)^{-2/5}\,, \tag{11}$$

where τ_0 is the cooling timescale and T_0 is the initial temperature. We can use our expression for the luminosity to find

$$L(t) = L_0\left(1+\frac{5}{2}\frac{t}{\tau_0}\right)^{-7/5}\,, \tag{12}$$

These equations tell us that the luminosity decays quickly at first, but slows its rate of change with time.

Over a very long time, a white dwarf will cool and its material will begin to crystallize, starting with the core. The star's low temperature means it will no longer emit significant heat or light, and it will become a cold black dwarf. Because the length of time it takes for a white dwarf to reach this state is calculated to be longer than the current age of the universe, it is thought that no black dwarfs yet exist. The oldest white dwarfs still radiate at temperatures of a few thousand kelvins. White dwarfs have an extremely small surface area to radiate this heat from, so they cool gradually, remaining hot for a long time.

Note that our book has an alternative derivation, copied below:
We can get an expression for the cooling rate because we know that the energy comes from the thermal energy of the ions in the core:

$$U = \frac{3}{2}RMT_c \tag{13}$$

$$L = -\frac{dU}{dt} = -\frac{3}{2}\frac{R}{\mu}M\frac{dT_c}{dt} = -\frac{3}{7}\frac{R}{\mu}M\frac{T_c}{L}\frac{dL}{dt} \tag{14}$$

After some algebra:

$$-\frac{dL}{dt} \propto MT_c^6\,. \tag{15}$$

So the cooling decreases sharply with the decreasing temperature. A low mass (cooler) white dwarf evolves slowly.

Neutron Stars

Stars with masses $> 8 - 10\ M_\odot$ can produce neutron stars instead of white dwarfs. The electron degeneracy that creates white dwarfs is not strong enough to support the increased mass, leading to neutron degeneracy (neutrons are also fermions). The process of creating neutron stars is that of electron capture. At high densities ($\sim 1 \times 10^{10}\,\mathrm{kg\,m^{-3}}$), the low energy arrangement allows for

$$p^+ + e^- \to n + \nu_e \tag{16}$$

Thus, at high densities, the stellar remnant is entirely neutrons.

Neutron star radii

Neutron stars are (barely) nonrelativistic, so $\gamma = 5/3$ and $n = 3/2$. Using the relationship between central pressure and density appropriate to polytropes,

$$P_c = (4\pi)^{1/3} B_n G M^{2/3} \rho_c^{4/3}, \tag{17}$$

Combining this with the pressure-density relation for a degenerate neutron gas, we get

$$K_1 \rho^{5/3} = (4\pi)^{1/3} B_n G M^{2/3} \rho_c^{4/3} \tag{18}$$

After some manipulation, and using values for $n = 3/2$ polytrope constants, we find

$$R = 14 \left(\frac{M}{1.4 M\odot} \right)^{-1/3} km. \tag{19}$$

This R is only slightly higher than what we get using more sophisticated models (~10 km) for neutron stars that have had a chance to cool off from their initial formation and become fully degenerate.

The slight discrepancy is due to several reasons: (i) the fluid is not purely neutrons; there are some protons too, which do not contribute to the neutron degeneracy pressure, (ii) the neutrons are not too far from being relativistic and this reduces their pressure compared to the fully non-relativistic pressure we have used above, and (iii) the neutron matter also has a considerably more complex structure than a simple degenerate electron gas due to the nuclear forces between the neutrons.

Just like WDs, neutron stars follow the mass-volume relationship: $MV = constant$. Just like WDs, they also have a mass limit. For NSs, this limit is 2.2 $M_\odot$ for non-rotating and 2.9 $M_\odot$ for rotating. Collapse of a neutron star leads to a black hole.

NS Rotation

Neutron stars are rapidly rotating, due to conservation of angular momentum during collapse[1]. Angular momentum is $L = I\omega$, so we can write

$$I_i\omega_i = I_f\omega_f \,. \tag{20}$$

The moment of inertia $I = CMR^2$ where C is a constant that depends on the geometry ($C = 2/5$ for a sphere). Thus, assuming the mass is unchanged.

$$R_i^2\omega_i = R_f^2\omega_f \,. \tag{21}$$

$$\omega_f = \omega_i \left(\frac{R_i}{R_f}\right)^2 \,. \tag{22}$$

But what is the initial radius? It's the radius of the core, which is basically the radius of a WD. Dividing our two expressions for the radius of NS and WDs, we find

$$\frac{R_{\rm core}}{R_{\rm ns}} \approx 500 \tag{23}$$

Thus, a NS is spinning 10^4 times faster than its progenitor! That's a lot!

Magnetic Fields

Another important property of neutron stars is that they are superconductors, i.e., they have nearly infinite electrical conductivity. Therefore, electric currents flow with essentially no resistance and magnetic fields diffuse very little. Therefore, the magnetic field within them is said to be "frozen into the fluid" meaning that any field line that passes through a given fluid element is trapped in that fluid element and moves and deforms with it.

Magnetic flux must be conserved during collapse

$$\Phi = \int_S B \cdot dA \,, \tag{24}$$

where A is the surface area. Assuming a sphere, $A = 4\pi R^2$ and we therefore have

$$B4\pi R_i^2 = B4\pi R_f^2 \tag{25}$$

$$B_{ns} = B_i \left(\frac{R_i}{R_{ns}}\right)^2 \,, \tag{26}$$

which is the same factor as we had previously. The magnetic field must also increase $\sim 10^4$ times.

[1] There is some disagreement on this point! But the derivation is nice and seems to give the right answer

NS Temperatures

NS are hot! They begin their lives at $\sim 10^{11}$ K and like WDs cool slowly as they age. Using this temperature and a radius of $\sim$ 10 km, Stephan-Boltzmann gives $L = 10^{26}$ W, which is similar to that of the Sun. From Wien's law, the peak is near 3 cm, in the X-ray regime.

Pulsars!

Neutron stars emit radiation, and this radiation can be beamed toward us. In such cases, we get one "pulse" of radiation each rotation. Such objects are called "pulsars." Pulsars were first detected by Joclyn Bell, then a grad student, and her advisor Anthony Hewish in 1967. She noticed a "scruff' in her chart record that had a regular period. She initially thought it may be from aliens, but after finding more sources of scruff, she realized that there were many such pulsars in the sky, and the alien hypothesis was not likely. This discovery led to a nobel prize (for Hewish :( ).

There are a few thousand pulsars known, and the next generation of radio telescopes will undoubtedly find many multiples more (FAST in China is finding tons). Pulsars have periods ranging from seconds to milliseconds. Because they are so regular in their pulses, we can accurately measure their periods, and also period derivatives $\dot{P}$. Pulsar lifetimes can be expressed as $P/\dot{P}$ - a characteristic value is $10^7 - 10^8$ years.

Pulsars generally have periods of between 0.25 and 2 s, with $\dot{P}$ on the order of 10^{-15} s.

How do we know that pulsars are small? Let's balance centripetal acceleration with gravity. If centripetal acceleration were larger than the gravitational acceleration, the object would fly apart.

$$\omega^2 R = G\frac{M}{R^2} \tag{27}$$

Since $P = 2\pi\omega$, this expression solves to

$$P = 2\pi\sqrt{\frac{R^3}{GM}}\,. \tag{28}$$

or

$$R = \left(\frac{P^2 GM}{4\pi^2}\right)^{1/3}\,. \tag{29}$$

If we take $P = 1\,s$, I find $R =\simeq 10^4$ km. The fastest pulsars, however, rotate with $P \simeq 0.001$ s, which gives ~ 100 km, which is way too small for a WD.

Three distinct classes of pulsars are currently known to astronomers, according to the source of the power of the electromagnetic radiation:

- rotation-powered pulsars, where the loss of rotational energy of the star provides the power,

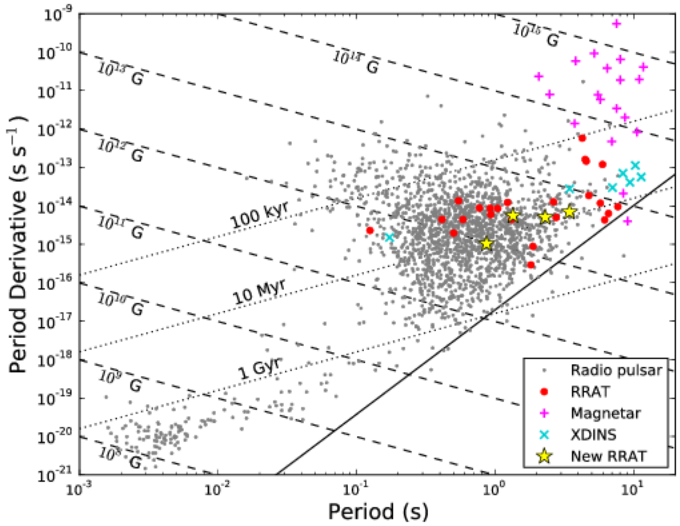


Figure 3: The $P - \dot{P}$ diagram for pulsars.

- accretion-powered pulsars (accounting for most but not all X-ray pulsars), where the gravitational potential energy of accreted matter is the power source (producing X-rays that are observable from the Earth),
- magnetars, where the decay of an extremely strong magnetic field provides the electromagnetic power.

For rotation-powered pulsars we can calculate the rate of energy loss. The rotational energy is:

$$K = 1/2I\omega^2 = \frac{2\pi^2 I}{P^2} \tag{30}$$

and therefore energy is lost as

$$\frac{dK}{dt} = -\frac{4\pi^2 I\dot{P}}{P^3} \tag{31}$$

The term $-dK/dt$ is the luminosity. If we assume $P = 1\,\mathrm{s}$, $\dot{P} = 10^{-15}\,\mathrm{s}$, and $I = 2/5MR^2 = 2/5 \times 1\ M_\odot \times (10\,\mathrm{km})^2$, I get $3 \times 10^{24}\,\mathrm{W}$.

Pulsar emission is still poorly understood, although we have plenty of theories. The radiation does imply an increase in spin period though, and a positive value of $\dot{P}$. When a pulsar's spin period slows down sufficiently, the radio pulsar mechanism is believed to turn off (the so-called "death line"). This turn-off seems to take place after about 10-100 million years, which means of all the neutron stars born in the 13.6-billion-year age of the universe, around 99% no longer pulsate.

Given the dispersion relation

$$\omega^2 = k^2c^2 + \omega_p^2. \tag{32}$$

where the angular plasma frequency is

$$\omega_p^2 \equiv \frac{4\pi n_e e^2}{m_e} \tag{33}$$

The plasma frequency is related to the electron density as

$$\nu_p = \frac{\omega_p}{2\pi} \approx 8.979 \text{ kHz } \left(\frac{n_e}{\text{cm}^{-3}}\right)^{1/2}. \tag{34}$$

The propagation speed is given by the group velocity

$$\begin{aligned}
v_g \equiv \frac{\partial \omega}{\partial k} &= \frac{\partial}{\partial k}\omega & (35)\\
&= \frac{\partial}{\partial k}\left(k^2c^2 + \omega_p^2\right)^{1/2} & (36)\\
&= \frac{\cancel{2}kc^2}{\cancel{2}\left(k^2c^2 + \omega_p^2\right)^{1/2}} & (37)\\
&= \frac{\frac{\cancel{1}}{\cancel{c}}\left(\omega^2 - \omega_p^2\right)^{1/2} c^{\cancel{2}}}{\omega} & (38)\\
&= c\left(1 - \frac{\omega_p^2}{\omega^2}\right)^{1/2} & (39)\\
&= c\left(1 - \frac{\nu_p^2}{\nu^2}\right)^{1/2} & (40)\\
&\equiv c\mu & (41)
\end{aligned}$$

where $\mu \leq 1$ is the index of refraction. Below the plasma frequency, μ is imaginary and the waves cannot propagate. For the ionosphere, the electron density peaks at about 10^6 cm^{-3} and so the plasma frequency is about 9 MHz. In the ISM, for $n_e \sim 0.1$ cm^{-3}, the plasma frequency is about 3 kHz.

Dispersive Time Delay

The total propagation time as a function of path length through the medium is

$$\begin{aligned}
t_{\text{total}} &= \int_0^D \frac{dl}{v_g} & (42)\\
&= \int_0^D \frac{dl}{c}\left(1 - \frac{\nu_p^2}{\nu^2}\right)^{-1/2} & (43)\\
&\approx \int_0^D \frac{dl}{c}\left(1 + \frac{\nu_p^2}{2\nu^2}\right) & (44)\\
&= \int_0^D \frac{dl}{c} + \int_0^D \frac{dl}{c}\frac{\nu_p^2}{2\nu^2} & (45)\\
&= \frac{D}{c} + \frac{e^2}{2\pi m_e c}\frac{\int_0^D n_e(l)dl}{\nu^2} & (46)\\
&= t_{\text{geometric}} + t_{\text{dispersive}}. & (47)
\end{aligned}$$

where in the last step we broke up the total time into the geometric travel time and the dispersive delay. Therefore,

$$
\begin{aligned}
t_{\text{dispersive}} &= \frac{e^2}{2\pi m_e c}\frac{\int_0^D n_e(l)dl}{\nu^2} && (48)\\
&\equiv K\frac{\text{DM}}{\nu^2} && (49)\\
&\approx 4.149 \text{ ms} \left(\frac{\text{DM}}{\text{pc cm}^{-3}}\right)\left(\frac{\nu}{\text{GHz}}\right)^{-2} && (50)
\end{aligned}
$$

where $\text{DM} \equiv \int_0^D n_e(l)dl$ is the dispersion measure and $K \equiv \frac{e^2}{2\pi m_e c} \approx 4.149 \text{ ms GHz}^2 \text{ pc}^{-1} \text{ cm}^3$ is the dispersion constant.

Pulsar Utility

Pulsars have proven themselves to be incredibly useful objects for for studying the interstellar medium and for testing concepts in physics.

The radiation from pulsars passes through the interstellar medium (ISM) before reaching Earth. Free electrons in the warm (8000 K), ionized component of the ISM and H II regions affect the radiation in by introducing a frequency-depending delay in the pulse arrival times.

Because of the dispersive nature of the interstellar plasma, lower-frequency radio waves travel through the medium slower than higher-frequency radio waves. The resulting delay in the arrival of pulses at a range of frequencies is directly measurable as the dispersion measure of the pulsar. The dispersion measure is the total column density of free electrons between the observer and the pulsar:

$$DM = \int n_e d\ell \tag{51}$$

where n_e is the electron density of the ISM and the integration is along the path. The dispersion measure is used to construct models of the free electron distribution in the Milky Way.

Pulsars have also been used to detect the so-called stochastic background of gravitational waves. This signal is produced from the combined effects of all supermassive black holes in the Universe. There are 3 consortia around the world which use pulsars to search for gravitational waves. In Europe, there is the European Pulsar Timing Array (EPTA); there is the Parkes Pulsar Timing Array (PPTA) in Australia; and there is the North American Nanohertz Observatory for Gravitational Waves (NANOGrav) in Canada and the US. Together, the consortia form the International Pulsar Timing Array (IPTA). The pulses from Millisecond Pulsars (MSPs) are used as a system of Galactic clocks. Disturbances in the clocks will be measurable at Earth. A disturbance from a passing gravitational wave will have a particular signature across the ensemble of pulsars, and will be thus detected. NANOGrav just reported a detection of the gravitational wave background!

GR and Black Holes
C+O Chapter 17

The final endpoint of stellar evolution for us to discuss is black holes, which are the products of the most massive stars created. The treatment of black holes necessitates a discussion of general relativity, GR.

GR

In 1915, Einstein published his new theory of gravity, General Relativity (GR). At the time, Newtonian gravity was well-accepted, but there were hints that other physics remained unaccounted for. Foremost was the orbit of Mercury, for which the perihelion location (furthest distance from the Sun) shifts in a manner that cannot be explained by Newtonian gravity. Einstein in 1905 published his theory of special relativity, which reconciles Newton's laws of motion with electrodynamics. A new theory was needed to update gravity.

Equivalence principle

Special relativity tells us that the laws of physics are the same in all inertial reference frames. What is an inertial reference frame? A frame that is not accelerating! But, since gravity is an acceleration, and everything acts under the influence of gravity, can we really have inertial frames? We know that we do have inertial frames, so although gravity causes an acceleration, it must be different.

An observer in an accelerated reference frame must introduce what physicists call fictitious forces to account for the acceleration experienced by himself and objects around him. One example is the force pressing the driver of an accelerating car into his or her seat; another is the force you can feel pulling your arms up and out if you attempt to spin around like a top. Einstein's insight was that the pull of the Earth's gravitational field is fundamentally the same as these fictitious forces. The apparent magnitude of the fictitious forces always appears to be proportional to the mass of any object on which they act - for instance, the driver's seat exerts just enough force to accelerate the driver at the same rate as the car. By analogy, Einstein proposed that an object in a gravitational field should feel a gravitational force proportional to its mass, as embodied in Newton's law of gravitation.

A person in a free-falling elevator experiences weightlessness; objects either float motionless or drift at constant speed. Since everything in the elevator is falling together, no gravitational effect can be observed. In this way, the experiences of an observer in free fall are indistinguishable from those of an observer in deep space, far from any significant source of gravity. Such observers are the privileged ("inertial") observers Einstein described in his theory of special relativity: observers for whom light travels along straight lines at constant speed.

Gravity behaves differently. Einstein realized that if everyone were in a state of free-fall, we would have no method of knowing it! Also, since each spatial location feels a different force of gravity, it cannot be fully eliminated. Instead, Einstein proposed "local" reference frames where the acceleration due to gravity is essentially constant, which is known as the "Principle of Equivalence:" all local, free-falling nonrotating laboratories are fully equivalent for the performance of all physical experiments. Roughly speaking, the "equivalence principle" states that a person in a free-falling elevator cannot tell that they are in free fall.

We can see how odd this is by equating Newton's second law with the gravitational force:

$$ma = \frac{GMm}{r^2} \tag{52}$$

But the LHS shows an object's resistance to acceleration (its "inertial" mass) and the RHS gives the gravitational force, with m and M being the "gravitational charges." By why would these two masses be the same? Think of the situation for EM:

$$ma = \frac{qQ}{4\pi\epsilon_0 r^2}. \tag{53}$$

Strictly, we should change our nomenclature

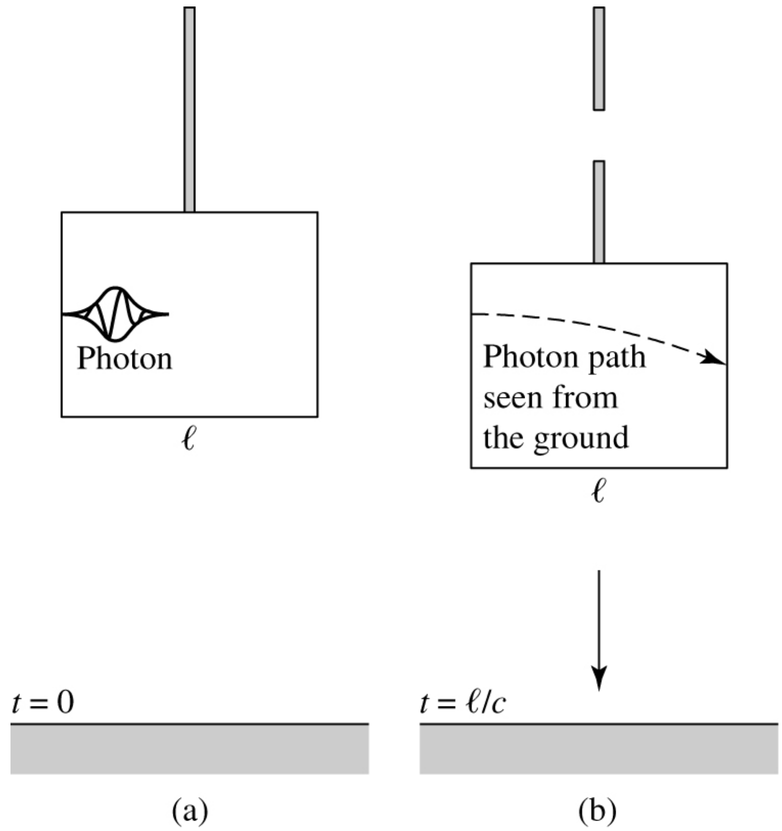
$$m_i a = \frac{GM_g m_g}{r^2} \tag{54}$$

Experiments have shown that m_i/m_g is unity to within one part in 10^{12}. This is known as the "weak equivalence principle."

Curved Light

Let's do another thought experiment! Imagine you have suspended a laboratory high above the earth. From one side of the lab you shoot a photon horizontal to the ground while at the same time releasing the lab so it is in free-fall. In the reference frame of the lab, the photon must maintain its horizontal displacement. An observer located on earth, however, would see the photon's path bend downward toward the Earth as the lab fell.

This deflection is minor, but measurable. The deflection of the photon is the quickest path to the other side of the lab, and evidence for curved spacetime.



(55)

Redshift

Redshift is a concept in astrophysics that describes the shifting of the frequency of light. We can define the redshift z as

$$z = \frac{\Delta\lambda}{\lambda_0} = \frac{\Delta\nu}{\nu_0}\,. \tag{56}$$

This is the definition, but we can relate it to the relative velocity via the Doppler effect:

$$z = \sqrt{\frac{1 + v_r/c}{1 - v_r/c}} - 1 \tag{57}$$

As long as $v_r \ll c$, we can make the expression

$$(1 + v_r/c)^{\pm 1/2} \simeq 1 \pm \frac{v_r}{2c}\,, \tag{58}$$

so for nonrelativistic motions

$$z = \frac{\Delta\lambda}{\lambda_0} \simeq \frac{v_r}{c} \tag{59}$$

By convention, we call decreases in wavelength (increases in frequency) blueshifts, for when the source of radiation is moving toward the observer. We call increases in wavelength (decreases in frequency) redshifts, for when the source of radiation is moving away from the observer.

Gravitational Redshift and Time Dilation

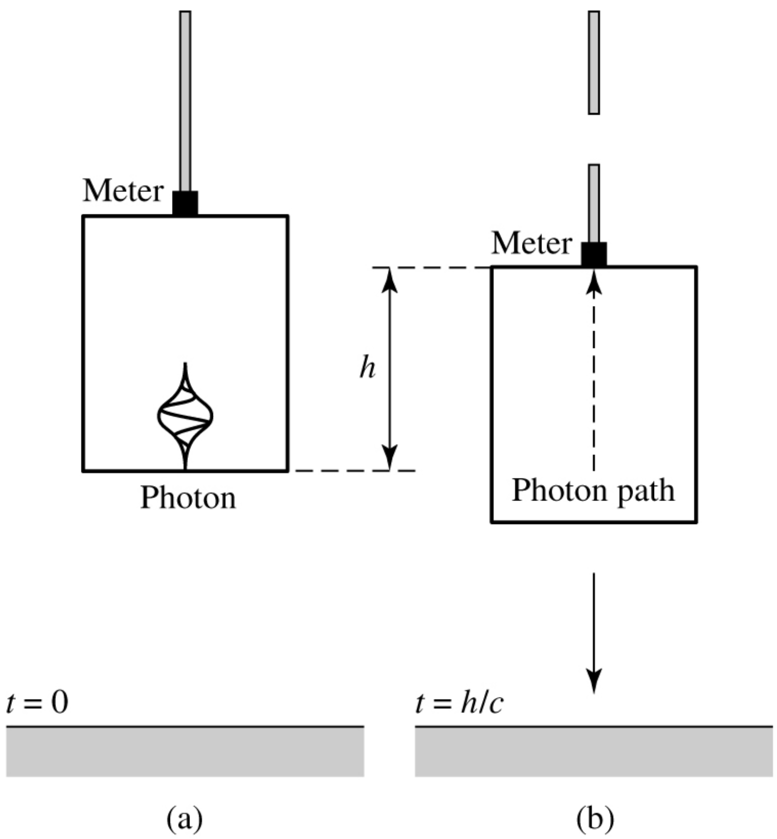
Let's again suspend a lab above the ground and cut the cable holding it just as we release a photon. This time, let's release the photon straight up from the lab's floor. Again, observers in the lab would not sense anything different with regards to the photon's motion.

From an external perspective, if the lab fell by distance h during the photon's travel, the ceiling is h closer to the photon than it was at the beginning. We would expect to measure a blueshift

$$\frac{\Delta\nu}{\nu_0} = \frac{v}{c} = \frac{gh}{c^2}\,. \tag{60}$$

But we know that such a shift is not measured. There must be something to oppose it, a "gravitational redshift" that applies when viewing accelerated frames. These frames can simply be due to gravity. Gravitational redshift is given by

$$\frac{\Delta\nu}{\nu_0} = -\frac{v}{c} = -\frac{gh}{c^2}\,. \tag{61}$$



(62)

The total gravitational redshift is given by integrating Equation 61 from r_0 to infinity. We must use $g = GM/r^2$ and set $h = dr$. A word of caution on this integration: the integration adds up contributions from different reference frames, but Equation 61 was derived from a local reference frame. The integration is only valid if spacetime is relatively flat.

$$\int_{\nu_0}^{\nu_\infty} \frac{d\nu}{\nu} \simeq -\int_{r_0}^{\infty} \frac{GM}{r^2c^2} dr\,, \tag{63}$$

which results in

$$\ln\left(\frac{\nu_\infty}{\nu_0}\right) \simeq -\frac{GM}{r_0c^2} \tag{64}$$

which is valid for relatively weak gravity and can be rewritten as

$$\frac{\nu_\infty}{\nu_0} \simeq e^{-GM/r_0c^2}\,. \tag{65}$$

$e^{-x} \simeq 1 - x$ if $x \ll 1$ so

$$\frac{\nu_\infty}{\nu_0} \simeq 1 - \frac{GM}{r_0c^2} \tag{66}$$

A more accurate derivation gives us

$$\frac{\nu_\infty}{\nu_0} \simeq \left(1 - \frac{2GM}{r_0c^2}\right)^{1/2}\,. \tag{67}$$

Let's put things in terms of redshift z:

$$z = \frac{\Delta\nu}{\nu_0} = \frac{\nu_0}{\nu_\infty} - 1 \tag{68}$$

$$= \left(1 - \frac{2GM}{r_0c^2}\right)^{-1/2} - 1 \tag{69}$$

$$\simeq \frac{GM}{r_0c^2} \tag{70}$$

Intervals and Geodesics

The heart of GR is Einstein's field equations. The derivation and application of these equations is beyond the scope of this course (and way beyond my understanding!). These equations relate the effect of mass (and energy!) on the curvature of spacetime. This is the basic tenet of GR: mass and energy curve spacetime.

Spacetime diagrams can help us to get an intuitive feel for these equations. The axes of spacetime diagrams can vary, but all have a measure of position on (at least) one axis and time, or *ct* on the other. On these diagrams, worldlines show the path of an object: straight up for stationary objects, diagonal for constant velocity, more complicated for more complicated motion (but always moving upwards).

Light obviously moves at a constant velocity, and so should move as a straight line in a spacetime diagram. We can define a "lightcone" emanating from a single point in spacetime that gives all possible spacetime paths for a photon. Running time backwards gives all past possible paths. This leaves wide areas of spacetime that are not accessible! The finite speed of light means that not all of spacetime is accessible.

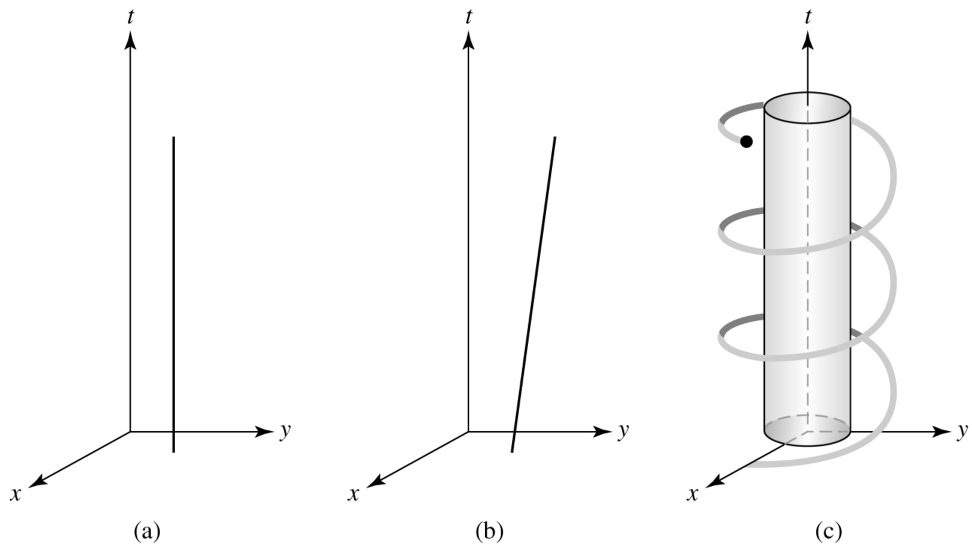


Figure 4: Worldlines!

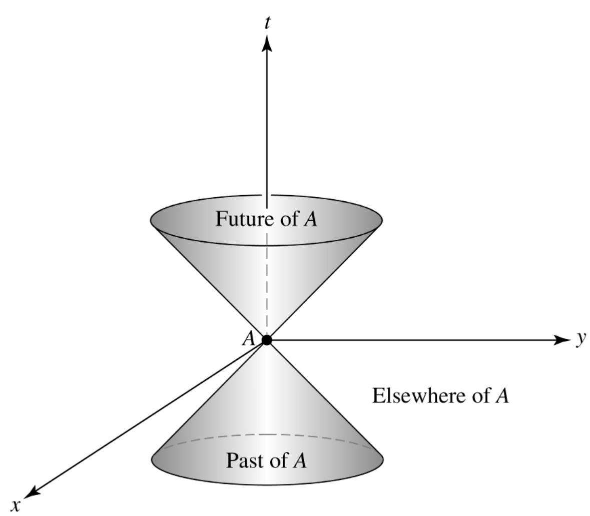


Figure 5: Light cones for photons.

Spacetime Intervals

We need to define a spacetime "distance." For cartesian distances of points specified as (x_1, y_1, z_1) and (x_2, y_2, z_2):

$$(\Delta\ell)^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \tag{71}$$

But in spacetime we have one more dimension, leading to (x_A, y_A, z_A, t_A) and (x_B, y_B, z_B, t_B):

$$(\Delta s)^2 = [c(t_B - t_a)]^2 - (x_B - x_A)^2 - (y_B - y_A)^2 - (z_B - z_A)^2 \,, \tag{72}$$

or the squared interval is the distance traveled by light squared minus the distance between events squared.

Note that $(\Delta s)^2$ can be positive, negative, or zero. If positive, the interval is timelike and light has enough time to travel between events A and B. We can then choose an inertial reference frame such that A and B happen at the same spatial location. Because the two events occur at the same place, the time measured between the two events is $\Delta s/c$. By definition, the time between the two events that occur at the same location is the proper time

$$\Delta\tau = \frac{\Delta s}{c} \tag{73}$$

The proper time is the elapsed time recorded by a watch moving along the worldline from A to B.

If $(\Delta s)^2 = 0$, we call it lightlike or "null." In this case, only light can go from A to B.

If $(\Delta s)^2 < 0$, the interval is spacelike and light cannot make the travel. Thus, nothing can make the travel. The lack of simultaneity in this situation means that there are inertial frames in which the time-ordering of the events is reversed, or where they occur at the same time. In the frame where the two events occur at the same time, we can define a proper distance:

$$\Delta L = \sqrt{-(\Delta s)^2} \,. \tag{74}$$

If a straight rod were connected between A and B, this would be the rest length of the rod.

We can define a "metric" as the differential distance along any (possibly curved) path:

$$(d\ell)^2 = (dx)^2 + (dy)^2 + (dz)^2 \tag{75}$$

Light will always follow the shortest possible path, but this need not be a straight line. We can integrate the above equation to find $\Delta\ell$.

Two events can be connected by infinitely many curved worldlines. We can then define a metric for flat spacetime:

$$(ds)^2 = (cdt)^2 - (d\ell)^2 = (cdt)^2 - (dx)^2 - (dy)^2 - (dz)^2 \tag{76}$$

As before, we can integrate this to find the total interval.

In flat spacetime, a straight timelike worldline between two events is a maximum; any other wordline between the same two events will not be straight and will have a smaller interval.

Geodesics

In a non-flat spacetime (one with mass), the straightest possible worldlines are curved. These are called "geodesics." The paths followed by freely falling objects are geodesics.

In curved spacetime, a geodesic is an extremum. For this chapter, the geodesics we will encounter are maxima.

Our overall goal here is to describe spacetime around a massive, spherical object. For this application, spherical coordinates are more useful:

$$(d\ell)^2 = (dr)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2 \tag{77}$$

$$(ds)^2 = (cdt)^2 - (dr)^2 - (rd\theta)^2 - (r\sin\theta d\phi)^2\,, \tag{78}$$

where the above expression is for flat spacetime. We need one that includes curvature. Note that the coordinates used here are those of an observer at rest a great ($\sim$ infinite) distance from the origin.

The derivation of the curved spacetime metric is way beyond the scope of this course. Karl Schwartzschild first solved Einstein's equations to get

$$(ds)^2 = \left(cdt\sqrt{1-2GM/rc^2}\right)^2 - \left(\frac{dr}{\sqrt{1-2GM/rc^2}}\right)^2 - (rd\theta)^2 - (r\sin\theta d\phi)^2\,, \tag{79}$$

Although the derivation of the Schwartzschild metric is beyond our reach, we can see that it has the expected properties. If we set $dt = 0$, a distance along a radial line (with $d\theta = d\phi = 0$) is the proper distance:

$$dL = \sqrt{-(ds)^2} = \frac{dr}{\sqrt{1-2GM/rc^2}}\,. \tag{80}$$

This tells us that the spatial distance dL is greater than the coordinate difference dr. This is the stretching of spacetime around massive objects. We can do the same thing with proper time when r is unchanging:

$$d\tau = \frac{ds}{c} = dt\sqrt{1-\frac{2GM}{rc^2}}\,. \tag{81}$$

Since $d\tau < dt$, time passes more slowly close to a massive object, as measured by an external observer.

The Orbit of a Satellite

Mass tells spacetime how to curve and spacetime tells mass how to move. Masses will follow the straightest possible worldline.

Let's say the satellite travels with $\omega = v/r$ entirely in the $\hat{\phi}$ direction, so $dr = d\theta = 0$ and

$d\phi = \omega dt$. We can input these into the Schwartzschild metric to find

$$(ds)^2 = \left[\left(cdt\sqrt{1 - 2GM/rc^2}\right)^2 - r^2\omega^2\right] dt^2 \tag{82}$$

$$= \left(c^2 - \frac{2GM}{c} - r^2\omega^2\right) dt^2 \tag{83}$$

We can integrate this expression over one orbit to get

$$\Delta s = \int_0^{2\pi\omega} \sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\, dt \tag{84}$$

The worldline must have a radial derivative of zero, so

$$\frac{d}{dr}(\Delta s) = \frac{d}{dr}\int_0^{2\pi\omega} \sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2}\, dt = 0 \tag{85}$$

$$\frac{d}{dr}\sqrt{c^2 - \frac{2GM}{r} - r^2\omega^2 dt} \tag{86}$$

$$\frac{2GM}{r^2} - 2r^2\omega^2 = 0 \tag{87}$$

$$v = r\omega = \sqrt{\frac{2GM}{r}} \tag{88}$$

This is the coordinate speed of the satellite for a circular orbit (the speed measured by a distant observer). Note that this is exactly what Newtonian gravity would predict!

The Schwartzschild Radius

You can compute the Schwartzschild radius R_S by equating the kinetic and potential energy of a photon. This (incorrect) derivation for the event horizon nevertheless gets the correct answer. We can prove this by taking a null worldline, $ds = 0$ for a photon falling straight into a black hole. Such a photon has $d\theta = d\phi = 0$, so

$$\frac{dr}{dt} = c\left(1 - \frac{2GM}{rc^2}\right) = c\left(1 - \frac{R_S}{r}\right) . \tag{89}$$

We see that the coordinate speed equals c at great distance, but as $r \to R_S$, $dr/dt \to 0$. We cannot get light inside the Schwartzschild radius. This is also called the "event horizon." Interior to this is the "singularity." Light is frozen at the event horizon. In fact the collapse of the star is frozen there too! We cannot see it because photons are trapped, but if they weren't we would see time standing still.

A Trip into a Black Hole

What happens as one falls into a black hole? Assume you take a trip to a black hole while shining a flashlight backwards from your direction of travel. Light from that flashlight will be more and more redshifted to an external observer as you call in. It will also grow dimmer to this same observer as time dilation increases the time between photons. But to you, all is well, at least for a while! Eventually, the gravitational force on your feet is much greater than that on your head (a tidal force). This is bad. You are eventually stretched out like spaghetti.

Hawking Radiation

It may feel like black holes are impossible forces, given that they are the end point of many stars' evolutionary paths. Stephen Hawking discovered, however, that black holes do "'evaporate" over long timescales. We know from quantum mechanics that the quantum world is filled with particles popping in an out of existence. This produces pairs of particles and their anti-equivalents. When these pairs recombine, they annihilate and all is well. Near a black hole however, one of the particles may fall into the event horizon, and the other may escape the system. This carries energy away from the black hole, leading to its evaporation.

The timescale is really long here!

$$t_{\rm evap} = 2650\pi^2 \left(\frac{2GM}{c^2}\right)^2 \left(\frac{M}{M_\odot}\right) \approx 2\times 10^{67} \left(\frac{M}{M_\odot}\right)\,{\rm yr}\,. \tag{90}$$

So normal black holes take a long time! Smaller black holes formed in the big bang primordial may be as small as 10^{-8} kg according to the Big Orange book, and so will be faster. But we aren't sure they exist....

Supernovae

A supernova is one of the most energetic events in the universe. The energy comes from the explosion of a star, but there are actually two main types: the explosion of a high-mass star at the end of its life and the explosion of a $\sim$ Solar mass white dwarf after accreting matter from a companion. We will cover the latter scenario when we deal with binaries. A SN leaves behind a neutron star or a black hole.

Famous SN

Compared to a star's entire history, a supernova is very brief (the explosion itself takes $\sim$minutes; the bright afterglow is visible for perhaps only a couple months).

From Wikipedia:

- The earliest possible recorded supernova, known as HB9, could have been viewed by unknown prehistoric people of the Indian subcontinent and then recorded on a rock carving, since found in Burzahama region in Kashmir and dated to 4500 ± 1000 BC.

- Later, SN 185 was documented by Chinese astronomers in AD 185.

- The brightest recorded supernova was SN 1006, which occurred in AD 1006 in the constellation of Lupus. This event was described by observers in China, Japan, Iraq, Egypt, and Europe.

- The widely observed supernova SN 1054 produced the Crab Nebula.

- Supernovae SN 1572 and SN 1604, the latest Milky Way supernovae to be observed with the naked eye, had a notable influence on the development of astronomy in Europe because they were used to argue against the Aristotelian idea that the universe beyond the Moon and planets was static and unchanging Johannes Kepler began observing SN 1604 at its peak on 17 October 1604, and continued to make estimates of its brightness until it faded from naked eye view a year later. It was the second supernova to be observed in a generation, after Tycho Brahe observed SN 1572 in Cassiopeia.

- There is some evidence that the youngest Galactic supernova, G1.9+0.3, occurred in the late 19th century, considerably more recently than Cassiopeia A from around 1680. Neither supernova was noted at the time. In the case of G1.9+0.3, high extinction from dust along the plane of our galaxy could have dimmed the event sufficiently for it to go unnoticed. The situation for Cassiopeia A is less clear; infrared light echos have been detected showing that it was not in a region of especially high extinction.

- The most famous SNR, however, is SN 1987A in the Large Magellanic Cloud. This SN went off in 1987 (thus the name) and is only 50 kpc distant. We have been observing it ever since. The progenitor was a blue supergiant. https://www.nasa.gov/feature/goddard/2017/the-dawn-of-a-new-era-for-supernova-1987a

We also detect SN in other galaxies, and these have proven key to our understanding of the Universe (most on this next semester if you take ASTR368).

Types of SN

Just like many things in astronomy, the designations of SNe are based on observed characteristics, and we later determined that this scheme was not intuitive. Oh well.

We have two primary tools for classifying SNs: light curves and spectra. Light curves are just the intensity as a function of time. Spectra can tell you which elements are in the SN explosion.

Type I SN have no H in their spectra. Type II do. The lack of H indicates that Type I SN come from stars that lack H envelopes.

Type Ia have strong Si II lines at 615 nm.
Type Ib SN have He lines.
Type Ic do not have Si or He lines.

Type I SN have light curves that reach maximum a few days after explosion. After maximum,

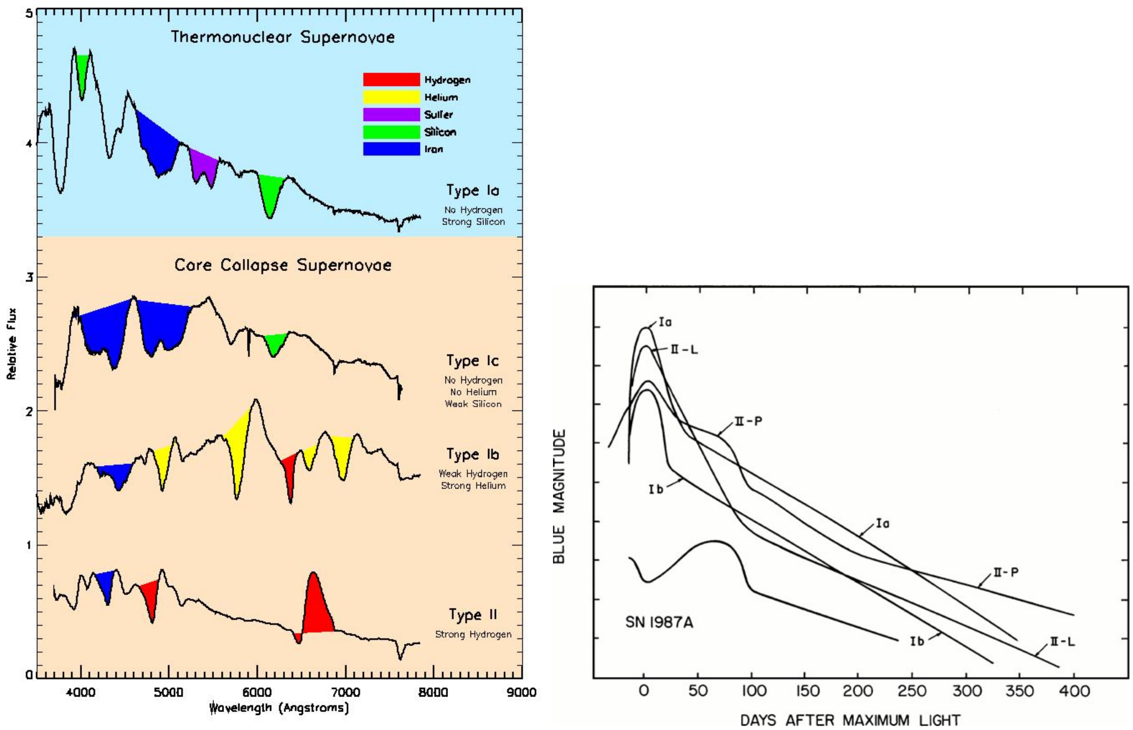


Figure 6: SN spectra and light curves.

they decline in brightness rapidly for 20 days, then decline slower for 50 days.

The lightcurves of Type II SNs are similar to those of Type Is, although Type II-Ps "plateau" and Type II-Ls are linear.

Confusingly, Type 1a are from white dwarfs. All others are core collapse of massive stars.

Runaway Fusion

A white dwarf star may accumulate sufficient material from a stellar companion to raise its core temperature enough to ignite carbon fusion, at which point it undergoes runaway nuclear fusion, completely disrupting it. There are three avenues by which this detonation is theorised to happen: stable accretion of material from a companion, the collision of two white dwarfs, or accretion that causes ignition in a shell that then ignites the core. The dominant mechanism by which type Ia supernovae are produced remains unclear. Despite this uncertainty in how type Ia supernovae are produced, type Ia supernovae have very uniform properties and are useful standard candles over intergalactic distances.

Core-Collapse SN

A typical Type II SN releases 10^{53} erg (10^{46} J) of energy. Interestingly, only about 1% of this goes into the energy of the ejected material, and 0.01% is released as photons. How does the

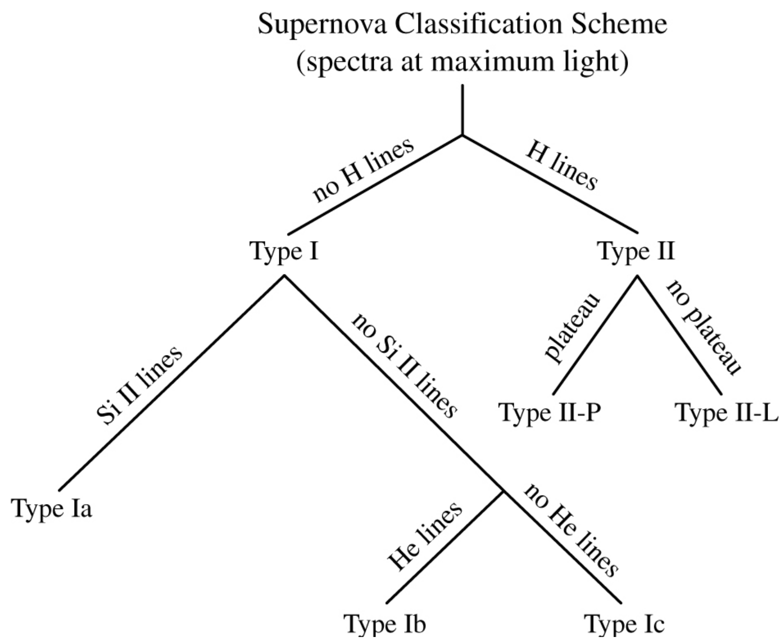


Figure 7: SN decision tree.

rest of the energy get out? Neutrinos!

As an interesting aside, SN1987A was first detected in neutrinos, before any photons from the explosion. This neutrino burst took place over 12.5 seconds. How could this have happened? The neutrinos travel at essentially the speed of light. The light was impeded by the dense shell around the progenitor, and therefore we had to wait until the shell became optically thin.

An iron core cannot release energy via fusion. In extremely high temperatures, iron can be gotten rid of through photodisinegration. This process strips iron down into protons and neutrons, and in the process absorbs energy in the form of photons. This is bad for the stability of the star. Furthermore, the protons themselves can capture electrons in the nearly 10^{10} K core, leading to the creation of neutrons and neutrinos:

$$p^+ + e^- \rightarrow n + \nu_e\,. \tag{91}$$

This releases tremendous energy.

Because of these endothermic reactions, and because the electrons are removed during electron capture, the star loses its support and rapidly collapses. During collapse, the outer layers are falling slower than the inner ones ($\tau \propto \rho^{-0.5}$), and when the collapse is progressing at the sound speed, the inner core decouples from the outer core. The outer core is now in free fall, suspended above a more rapidly collapsing (mostly iron) core. This probably won't end well.

The core has blown through its electron degeneracy pressure, but neutron degeneracy can resist the freefall (neutrons are also fermions, and obey the Pauli exclusion principle). The core rebounds slightly, and this sets up a shock wave. The details of what happens next depend on the model, but in essence the shock wave must travel outwards to the stellar surface, blowing those layers into the local medium.

What is left behind depends on the mass of the star. The most massive stars will become black holes, whereas less massive ones will become neutron stars. Your book notes that the cutoff mass is about 25 $M_\odot$.

Radioactive Decay

The chaotic fusion processes in the last few moments of a star's life lead to the creation of numerous radioactive isotopes. Your book notes $^{57}_{27}$Co and $^{44}_{22}$Ti, for example. Many of these decay through beta decay, releasing an electron, a neutrino electron, and radiation. These elements provide clocks that astronomers can use, since the decay will proceed as an exponential decay:

$$N(t) = N\_0e^{-\lambda t} \tag{92}$$

Nucleosynthesis within SNe

SNe also create elements and this is how the Universe can make all elements more massive than iron (neutron star collisions may also play a large role).

Fusion reactions are impeded by the high Coulomb barrier, but neutrons can penetrate this barrier and hence initiate fusion reactions. For example:

$$^A_Z X + n \rightarrow^{A+1}_Z X + \gamma\,. \tag{93}$$

If beta decay is slow compared to neutron capture this is called the rapid or "r-process." If the decay is fast compared to neutron capture, it's called the slow or "s-process." We can have s-process during normal stellar evolution, but r-process can only occur in a SN when large neutrino fluxes exist. The end result of the s- and r-processes is neutron-enriched elements.

Gamma Ray Bursts

Gamma-ray bursts (GRBs) are extremely energetic explosions that have been observed in distant galaxies. They are the brightest electromagnetic events known to occur in the universe. Bursts can last from ten milliseconds to several hours. After an initial flash of gamma rays, a longer-lived "afterglow" is usually emitted at longer wavelengths (X-ray, ultraviolet, optical, infrared, microwave and radio).

The intense radiation of most observed GRBs is thought to be released during a supernova or superluminous supernova as a high-mass star implodes to form a neutron star or a black hole. A subclass of GRBs (the "short" bursts) appear to originate from the merger of binary neutron stars.

The sources of most GRBs are billions of light years away from Earth, implying that the explosions are both extremely energetic (a typical burst releases as much energy in a few seconds as the Sun will in its entire 10-billion-year lifetime) and extremely rare (a few per galaxy per million years). All observed GRBs have originated from outside the Milky Way. It has been hypothesized that a gamma-ray burst in the Milky Way, pointing directly towards the Earth, could cause a mass extinction event.

GRBs were first detected in 1967 by the Vela satellites, which had been designed to detect covert nuclear weapons tests; this was declassified and published in 1973. In 1997 the first X-ray and optical afterglows were detected from a GRB and direct measurement of their redshifts using optical spectroscopy, and thus their distances and energy outputs. These discoveries, and subsequent studies of the galaxies and supernovae associated with the bursts, clarified the distance and luminosity of GRBs, definitively placing them in distant galaxies.

Cosmic Rays

SN also produce cosmic rays, charged particles that travel through space at incredible speeds.The Sun also produces cosmic rays, although Solar cosmic rays particles are comparably low energy.

Because they are charged, cosmic rays interact with magnetic fields.

Cosmic rays are responsible for heating in the interstellar medium. If fact, they are the only source of heating that can penetrate dense molecular clouds.

Cosmic rays are a serious impediment to long-distance human travel. The Earth is surrounded by its magnetic field, which diverts cosmic rays around it. Once astronauts leave this protected region, they can be exposed to cosmic rays, which can damage their internal organs.

1 Shocks! (Draine Ch. 35+36) and Supernovae

The folowing notes are adapted from my ISM course. They use the textbook by Bruce Draine.

Shocks are created when a flow moves faster than the local sound speed. What can do this?
–novae and supernovae
–fast stellar winds
–expanding H II regions
–gas falling into the potential of spiral arms
–colliding interstellar clouds

1.1 The Sound Speed

What is the sound speed? In an unmagnetized gas,

$$c_s = \left(\frac{\gamma P}{\rho}\right)^{0.5}, \tag{94}$$

where γ is the adiabatic index. γ takes values of $\gamma = 5/3$ for ideal monotonic gas, $\gamma = 7/5$ for diatomic gas. Partially ionized gas can take values between these two extremes, but fully ionized gas has $\gamma = 5/3$. We almost always assume an ideal gas, so $P = nkT$ and therefore

$$c_s = \left(\frac{\gamma kT}{\mu m_H}\right)^{0.5}, \tag{95}$$

where μ is again the mean particle mass. Let's compute c_s for the various phases of the ISM!

For magnetized gas, we add in the magnetic pressure $P_B = B^2/4\pi$ so

$$c_{ms} = \left(\frac{\gamma P}{\rho} + \frac{B^2}{4\pi\rho}\right)^{0.5}, \tag{96}$$

where c_{ms} is the magnetosonic speed.

The ratio of the speed of the flow to the speed of sound in the gas is the Mach number in honor of Ernst Mach, a late 19th century physicist who studied gas dynamics. "Subsonic" conditions occur for Mach numbers less than one, $M < 1$. As the speed of the object approaches the speed of sound, the Mach number is nearly equal to one, $M \simeq 1$, and the flow is said to be "transonic." Supersonic conditions occur for Mach numbers greater than one, $M > 1$. Sometimes, you may hear "hypersonic," which is $M > 5$.

A shock separates $M = 0$ (the ambient medium) from $M > 1$. The shock dissipates heat, and therefore due to the entropy generation it is irreversible.

2 The Fluid Equations

The ISM is a fluid, and so we must treat its physics using the fluid equations. Fluid mechanics in the ISM have three main useful equations: the continuity equation, the momentum equation, and the energy equation. These are collectively known as the "conservation equations," or sometimes simply as the "fluid equations." The names clue us in to the quantity that is conserved. We will eventually add in a fourth equation from Maxwell's Laws.

2.1 Mass Conservation

The (mass) continuity equation conserves mass. Assume we have some comoving volume $\Omega(t)$ (apologies, I'm following Draine's notation here). From mass conservation, $\rho\Omega =$ constant. Therefore,

$$\frac{\partial}{\partial t}(\rho\Omega) = \rho\frac{\partial\Omega}{\partial t} + \Omega\frac{\partial\rho}{\partial t} = 0\,. \tag{97}$$

We can solve this to find

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0\,. \tag{98}$$

This equation is telling us that the change in density with time (the first term) must be balanced by the divergence in the quantity $\rho\vec{v}$, which is the mass flux. Any change in density must be balanced by a changing mass flow rate - either changing the flow speed or the flow density.

2.2 Conservation of Momentum

The momentum equation conserves momentum (duh!), and is another way of stating $F = ma$. Again, let's consider only one dimension. What forces are acting on the fluid? First, there are "body forces" that act at a distance: the electric, magnetic, and gravitational fields. Second there are "surface forces:" the pressure acting on the surface and the shear force.

We can write $m\vec{a}$ as $\rho\Omega\frac{\partial}{\partial t}\vec{v}$ and then equate that to all forces in the system:

$$\rho\Omega\frac{\partial}{\partial t}\vec{v} = \vec{F}_{\rm pressure} + \vec{F}_{\rm EM} + \vec{F}_{\rm gravity} + \vec{F}_{\rm viscosity} \tag{99}$$

If we have some surface element $d\vec{S}$ outward. Then, the external fluid presses inward so that the net pressure on the fluid is

$$\vec{F}_{\rm pressure} = \int (-PdS) = \int -\nabla P d\Omega\,, \tag{100}$$

and by Gauss's theorem,

$$\vec{F}_{\rm pressure} = -\Omega\nabla P\,. \tag{101}$$

For gravity,

$$\vec{F}_{\rm gravity} = (\rho\Omega)(-\nabla\Phi_{\rm gravity}) \;= \rho\Omega g, . \tag{102}$$

This is Poisson's law, albeit in a form that is probably a bit unfamiliar.

Putting it all together, after the derivation of $\vec{F}_{\rm EM}$ (which is a little involved) and the viscosity (a little confusing, but we'll ignore it later anyway), we find

$$\rho\frac{D\vec{v}}{Dt} = -\nabla\left(P + \frac{B^2}{8\pi}\right) + \frac{1}{4\pi}(\vec{B}\cdot\nabla)\vec{B} - \rho g + \hat{x}_i\frac{\partial}{\partial x_j}\sigma_{ij}\,, \tag{103}$$

where the term $\hat{x}_i\frac{\partial}{\partial x_j}\sigma_{ij}$ is the viscosity, which we will ignore.

A simplified form of Equation 103 is the case when $\vec{B} = 0$ and $g = 0$. This is then known as a "Navier-Stokes" equation.

2.3 Conservation of Energy

Conservation of energy is more complicated. The mechanical power or mechanical work (dE/dt) is just the pressure times the change in volume. The change in volume is just $dV = dSv$. Thus, we can integrate the momentum equation over the surface times v to find the mechanical work.

There is also heating work, which is the difference of the heating Γ and the cooling Λ.

Draine derives the expression, but it's ugly and doesn't need to be reproduced here.

3 The Rankine-Huginot "Jump" Conditions

Now that we have our fluid equations, we can look at the physics of shocks. We are going to be in the frame of the shock. Although the shock is propagating into the ISM, the shock frame is stationary.

If there is a discontinuity (a shock) the mass, momentum, and energy must be conserved. We can therefore set up pre- and post-shock conditions, called the "Rankine-Huginot" conditions.

We are going to be making two simplifications: first, that the flow is "steady," so that $\partial/\partial t = 0$; second, that the shock is "plane-parallel" so that the flow is entirely in $\hat{x}$ ($\partial/\partial y = \partial/\partial z = 0$). Finally, we further assume that there is a "single species," so the flow of all particles has the same velocity.

3.1 Conservation of Mass

If the flow is constant or "steady," $\partial\rho/\partial t = 0$. If the flow is in one dimension, we can then write:

$$\rho\vec{v} = \text{constant}\,. \tag{104}$$

The mass flux is a conserved quantity.

3.2 Conservation of Momentum

We can integrate Equation 103 and simplify it using the mass conservation equation and ignoring viscous forces arrive at

$$\rho v_x^2 + P + \frac{B_y^2 + B_z^2}{8\pi} = \text{constant}\,, \tag{105}$$

where ρv^2 is the "ram pressure," or the pressure exerted by a flow, and $\frac{B_y^2+B_z^2}{8\pi} = \frac{1}{8\pi}B_\perp^2$ is the magnetic pressure perpendicular to the flow.

3.3 Conservation of Energy

Ignoring viscosity again, the energy conservation equation reduces to

$$\left[\frac{\rho v^2}{2}+\frac{\gamma P}{(\gamma-1)}\right]v_x+\frac{B_y^2+B_z^2}{8\pi}v_x-\frac{B_yB_x+B_zB_x}{4\pi}v_x-\kappa\frac{dT}{dx}=\text{constant}\,, \tag{106}$$

where $\kappa\frac{dT}{dx}$ refers to the heating and cooling.

In the case of $B_x=0$ and $\kappa\frac{dT}{dx}=0$ (no thermal conductivity), we find

$$\frac{\rho v^3}{2}+\frac{\gamma P}{(\gamma-1)}v+\frac{vB^2}{8\pi}=\text{constant}\,. \tag{107}$$

3.4 Conservation of Magnetic Flux

Although not from a fluid equation, we must also conserve magnetic flux. Maxwell tells us for infinite electrical conductance:

$$\frac{\partial\vec{B}}{\partial t}=\nabla\times(\vec{v}\times\vec{B})\,. \tag{108}$$

If $\partial/\partial t=\partial/\partial x=\partial/\partial z=0$, and $B_x=0\,$,

$$vB=\text{constant} \tag{109}$$

3.5 Solutions to the Jump Conditions

We can now put everything together to determine how our variables of interest change post-shock. The quantity on the left hand side of the fluid equations must be the same pre and post shock, so for instance a change in density must be compensated by a change in some combination of velocity, pressure, or magnetic field strength. Let's assume that we know or can estimate the pre-shock quantities $\rho_1, P_1, v_1,$ and B_1, where "1" refers to pre-shock gas and we will use "2" for post-shock gas. We then have four equations and four unknowns.

We will follow the usual convention and replace v_x with u, so in the shock reference frame $v_s=u_1$. We can also define $x\equiv\rho_2/\rho_1$, so therefore from mass conservation $u_2=v_s/x$ and from magnetic flux conservation $B_2=xB_1$. The momentum and energy equations are then

$$\rho_1v_2^2+P_1+\frac{B_1^2}{8\pi} = \frac{\rho_1v_2^2}{x}+P_2+\frac{B_1^2}{8\pi}x^2 \tag{110}$$

$$\frac{1}{2}\rho_1v_s^3+\frac{\gamma}{\gamma-1}P_1v_s+\frac{B_1^2}{8\pi}v_s = \frac{1}{2}\frac{\rho_1v_s^3}{x^2}+\frac{\gamma}{\gamma-1}\frac{P_2v_s}{x}+\frac{B_1^2}{8\pi}v_sx \tag{111}$$

One solution to these equations is the trivial one: $\rho_1=\rho_2, u_1=u_2, P_1=P_2,$ and $B_1=B_2$. But this is boring.

We can solve the modified momentum equation for P_2 and substitute into the modified energy equation to eventually get a quadratic in x. Draine lists the solution then as

$$x = \frac{2(\gamma+1)}{D + \sqrt{D^2 + 4(\gamma+1)(2-\gamma)M_A^{-2}}}\,, \tag{112}$$

where

$$D \equiv (\gamma - 1) + 2M^{-2} + \gamma M_A^{-2} \tag{113}$$

$$M \equiv \frac{v_s}{\sqrt{\gamma P_1/\rho_1}} \tag{114}$$

$$M_A \equiv \frac{v_s}{B_1/\sqrt{4\pi\rho_1}}\,. \tag{115}$$

M is again the Mach number and M_A is the Alfven Mach number.

For a shock to exist, it must be supersonic, so $v_s > c_{ms}$ (Equation 96). We can then define yet another Mach number:

$$\mathcal{M} \equiv v_s/c_{ms}\,, \tag{116}$$

and this is the one that matters for a magnetized medium.

These equations don't reduce to a nice form unless we have a "strong shock:" $\mathcal{M} \gg 1$. In this case, $D \to (\gamma - 1)$, so

$$x \to \frac{\gamma+1}{\gamma-1} = 4 \text{ for } \gamma = 5/3\,. \tag{117}$$

It also follows that

$$u_2 \to \frac{\gamma-1}{\gamma+1} v_s = \frac{1}{4} v_s \text{ for } \gamma = 5/3\,. \tag{118}$$

If we then solve for the pressure P_2 and assume the ideal gas law $T_2 = P_2\mu/\rho_2 k$, then

$$T_2 \to \frac{2(\gamma-1)}{(\gamma+1)} \frac{\mu v_s^2}{k} = \frac{3}{16}\frac{\mu v_s^2}{k} \text{ for } \gamma = 5/3\,. \tag{119}$$

Draine provides handy values for T_2:

$$T_2 \approx 2890\,\mathrm{K} \left(\frac{\mu}{1.273 m_H}\right)\left(\frac{v_s}{10\ \mathrm{km\,s^{-1}}}\right)^2 \tag{120}$$

$$T_2 \approx 1.38 \times 10^7\,\mathrm{K} \left(\frac{\mu}{0.609 m_H}\right)\left(\frac{v_s}{1000\ \mathrm{km\,s^{-1}}}\right)^2, \tag{121}$$

where $\mu = 1.273 m_H$ for H I and $\mu = 0.609 m_H$ for fully ionized gas.

We can see these effects graphically in Draine Figure 36.1, for an unmagnetized flow.

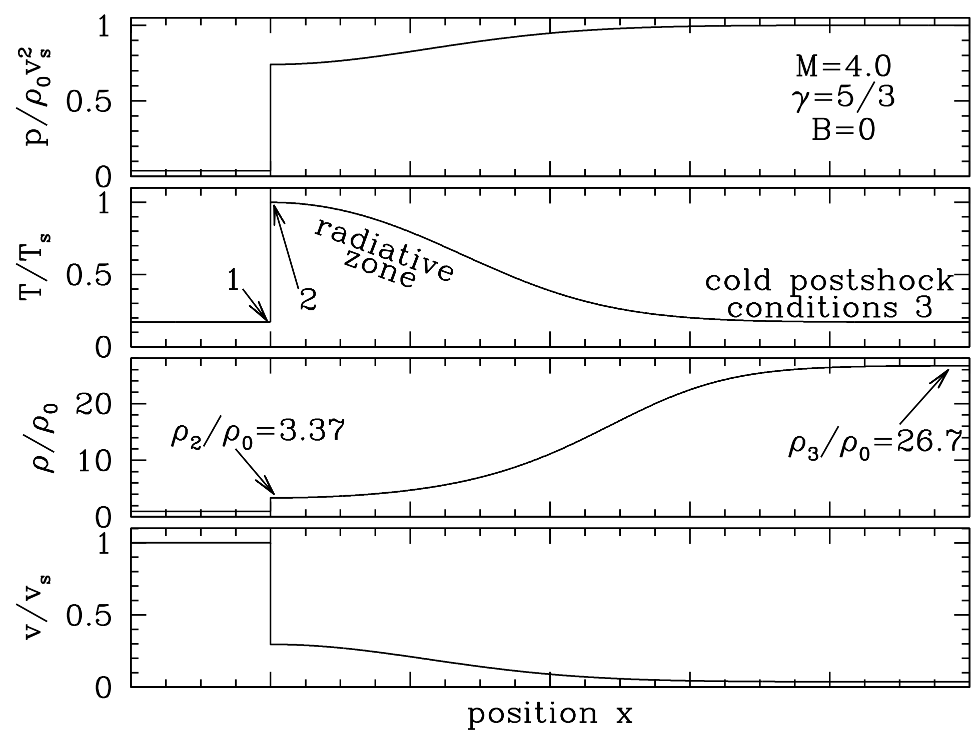


Figure 8: Draine figure 36.1 showing the structure of a nonmagnetic radiative shock with $M = 4$. Our treatment so far has only concerned positions 1 and 2.

4 Supernovae and the Three-Phase ISM

Supernovae (SNe) are super explosions of super stars at the end of their lifetimes. They are one of the most energetic phenomena in the Universe, and have a large impact on driving turbulence in the ISM. They are believed to be responsible for the acceleration of Galactic cosmic rays and the creation of the HIM.

As a review: SN Type Ia are from white dwarf accretion past the Chadrasehkar limit of $\sim 1.4\ M_\odot$. All other SN are from core-collapse, including the famous SN 1987A in the LMC.

Video of SN1987A:
https://www.youtube.com/watch?v=xigKhIfD_Ko
https://en.wikipedia.org/wiki/SN_1987A#/media/File:SN1987a_debris_evolution_animation_time_scaled.g

A typical SN has an energy of $E_0 = 10^{51}$ erg ($E_{51} = 1$), although some Type II SNe have $E_0 = 10^{52}$ erg. The ejected mass ($M_{\rm ej}$) ranges from $\sim 1.4\ M_\odot$ for Type I SNe, and $\sim 10 - 20\ M_\odot$ for Type II SNe.

We can understand SNe in terms of some very simple physics, and break things into three distinct phases: free expansion,

4.1 Phase I: Free Expansion

The first phase of expansion, the expansion energy is significantly greater than the energy contained in the local medium. For type II supernovae, the expansion initially moves into the exterior parts of the star.

Let's compare the ejecta energy with that of the explosion itself to see how fast this phase is propagating into the ISM:

$$\langle v_{ej}^2 \rangle = \left(\frac{2E_0}{M_{\rm ej}}\right)^{1/2} = 1.00 \times 10^4\ {\rm km\,s^{-1}}\ E_{51}^{1/2} \left(\frac{M_\odot}{M_{\rm ej}}\right)^{1/2}. \tag{122}$$

This is obviously much greater than the local sound speed of a few ${\rm km\,s^{-1}}$, which leads to a fast shock expanding into the ISM. Interior to the shock is the supernova remnant (SNR). As long as the material swept up by the shock is much less than the mass of the stellar ejecta, the expansion of the stellar ejecta proceeds at essentially a constant velocity equal to the initial shock wave speed, typically of the order of 10,000 ${\rm km\,s^{-1}}$. This is known as the "free expansion" phase and may last for approximately 200 years, at which point the shock wave has swept up as much interstellar material as the initial stellar ejecta. The supernova remnant at this time will be about 3 pc in radius.

Although the remnant is radiating thermal X-ray and synchrotron radiation across a broad range of the electromagnetic spectrum (from radio to X-rays), the initial energy of the shock wave will have diminished very little. Line emission from the radioactive isotopes generated

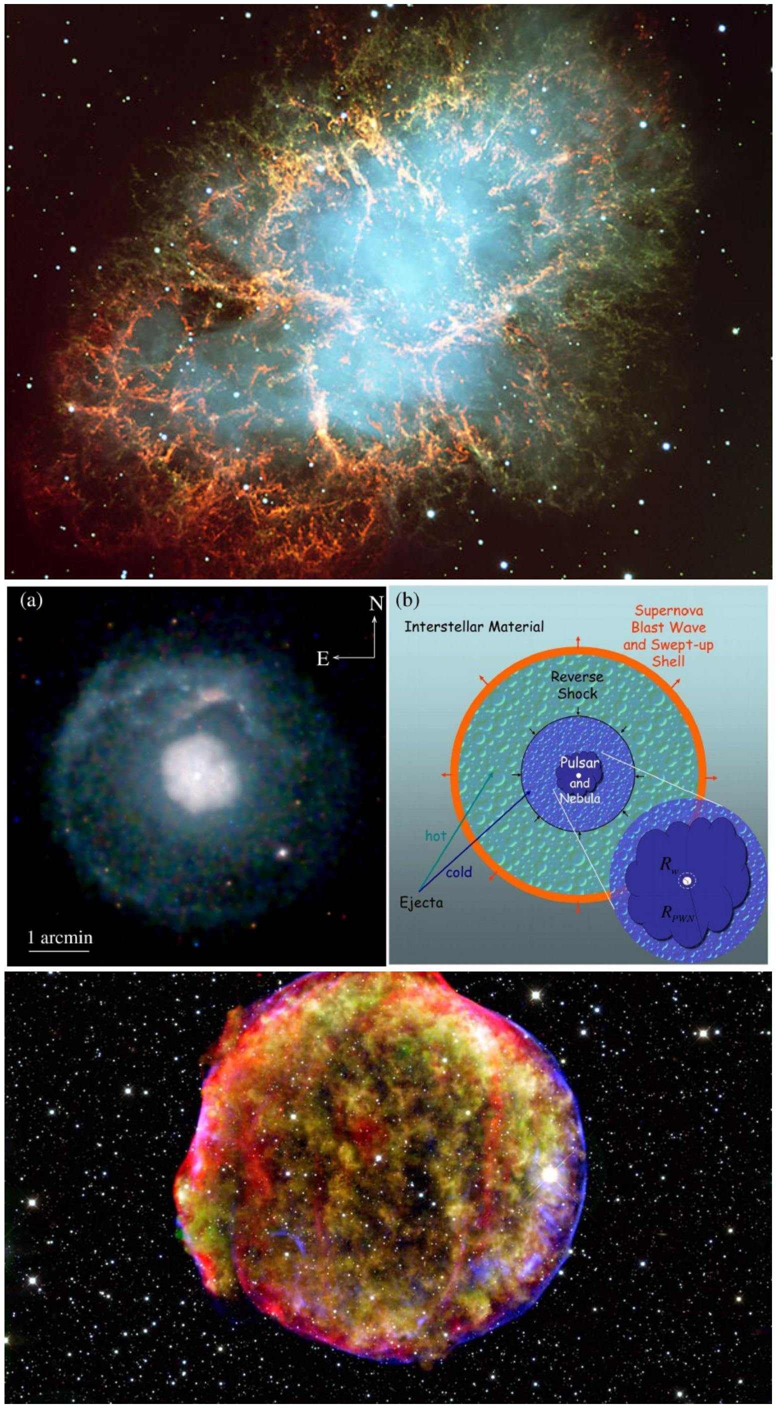


Figure 9: Crab nebula in the optical, diagram of blast wave and reverse shock, Tycho's SNR at X-ray wavelengths.

in the supernovae contribute significantly to the total apparent brightness of the remnant in the early years, but do not significantly affect the shock wave.

4.2 Phase II: Sedov-Taylor: The Blastwave™

As the remnant sweeps up ambient mass equal to the mass of the stellar ejecta, the wave will begin to slow and the remnant enters a phase known as adiabatic expansion, or the Sedov-Taylor or blast wave phase. The internal energy of the shock continues to be very large compared to radiation losses from thermal and synchrotron radiation, so the total energy remains nearly constant. The rate of expansion is determined solely the initial energy of the shock wave and the density of the interstellar medium.

As the density of the expanding ejecta drops (as T^{-3}), the pressure of the shocked gas behind the shock wave soon exceeds the thermal pressure in the ejecta. Because of this pressure difference, a reverse shock is created. There are now two shock fronts. The original one propagating outward is called the "blastwave" and the reverse shock propagating inward. The reverse shock re-heats the material in the SNR. [For the interested reader: Can we compute when the reverse shock is created?]

As the ejecta expands out from the star, it passes through the surrounding interstellar medium, heating it from 10^7 to 10^8 K, sufficient to separate electrons from their atoms and to generate thermal X-rays. The interstellar material is accelerated by the shock wave and will be propelled away from the supernova site at somewhat less than the shock wave's initial velocity. This makes for a thin expanding shell around the supernova site encasing a relatively low density interior.

This occurs at radius:

$$R_1 = \left(\frac{3M_{\rm ej}}{4\pi\rho_0}\right)^{1/3} , \tag{123}$$

when

$$t_1 \approx \frac{R_1}{\langle v_{\rm ej}^2\rangle^{1/2}} = 186\,{\rm yr}\left(\frac{M_{\rm ej}}{M_\odot}\right)^{5/6} E_{51}^{-1/2} n_0^{-1/3} \tag{124}$$

For $t \geq t_1$, the reverse shock has already reached the center of the SNR, and the entire SNR is hot. The remnant is still expanding due to the large pressure difference between the ISM and the SNR. It is emitting, but this radiation is not cooling the remnant significantly because the densities are low.

We can now approximate the expansion as a "point explosion" injecting energy E_0 into the uniform density ISM. We can neglect the finite mass of the ejecta (which is dwarfed by the mass of the swept up material), the radiative losses of energy (which are small compared to the energy of the system), and the pressure in the ambient medium (small compared to that in the SNR).

Here Draine switches to a strange dimensional analysis method to arrive at a classic result. We know that the shock radius R_s will expand at a rate dependent on the SN energy and the mass of the ISM:

$$R_s = AE^\alpha \rho^\beta t^\eta , \tag{125}$$

where the explosion occurs at $t = 0$. From dimensional analysis:

$$\begin{aligned} \text{Mass} : 0 &= \alpha + \beta && (126) \\ \text{Length} : 1 &= 2\alpha + 3\beta && (127) \\ \text{Time} : 0 &= -2\alpha + \eta \,. && (128) \end{aligned}$$

The mass condition arises because energy and mass must be proportional. To get length out, their exponents must differ only by a sign. The length condition arises because energy has in its units length^{-2} and density as length^{-3}; if $\alpha = -1$, $\beta = 1$ to get length out of the "equation." Similarly, for the third condition, energy has in its units time^{-2}, so if $\alpha = 1$, $\eta = 2$. These are not the final values though, just the proportions.

This leaves us with three equations and three unknowns. We can easily solve these to get $\alpha = 1/5, \beta = -1/5, \eta = 2/5$:

$$R_s = A\left(\frac{Et^2}{\rho_0}\right)^{1/5} , \tag{129}$$

where $A = 1.15$ from the exact solution. Neat! We can therefore rewrite our expansion terms, after realizing that $v^2 \propto T$:

$$\begin{aligned} R_s &= 1.52 \times 10^{19} \text{ cm}\, E_{51}^{1/5} n_0^{-1/5} t_3^{2/5} && (130) \\ v_s &= 1950 \text{ km s}^{-1}\, E_{51}^{1/5} n_0^{-1/5} t_3^{-3/5} && (131) \\ T_s &= 5.25 \times 10^{7} \text{ K}\, E_{51}^{2/5} n_0^{-2/5} t_3^{-6/5} , && (132) \end{aligned}$$

Or, the radius grows slowly with time, the shock velocity decreases slowly with time, and the temperature decreases with time.

Draine shows the formal Sedov-TailOr solution.

Draine mentions that the Sedov-Taylor solution is not too bad, although it does neglect some dynamical effects.

4.3 Phase III: Snowplow phase: Escape from Sedov-Taylor: The Reckoning: The Radiative Phase

[When does the Sedov-Taylor expansion phase end?] When radiative cooling becomes important. When temperatures cool to about $20,000$ K, ions and electrons begin recombining, the SNR leaves the Sedov-Taylor expansion phase.

This is probably a good time to talk about the radiation. The SNR is $\sim 10^7$ K. How does it radiate? X-rays and synchrotron primarily. Why not free-free? Well, there is free-free as well, but the synchrotron emission is much stronger.

After the temperature cools, the hot recombined electrons emit UV line radiation. This is much more efficient at cooling the remnant. For a cooling function Λ that has units of

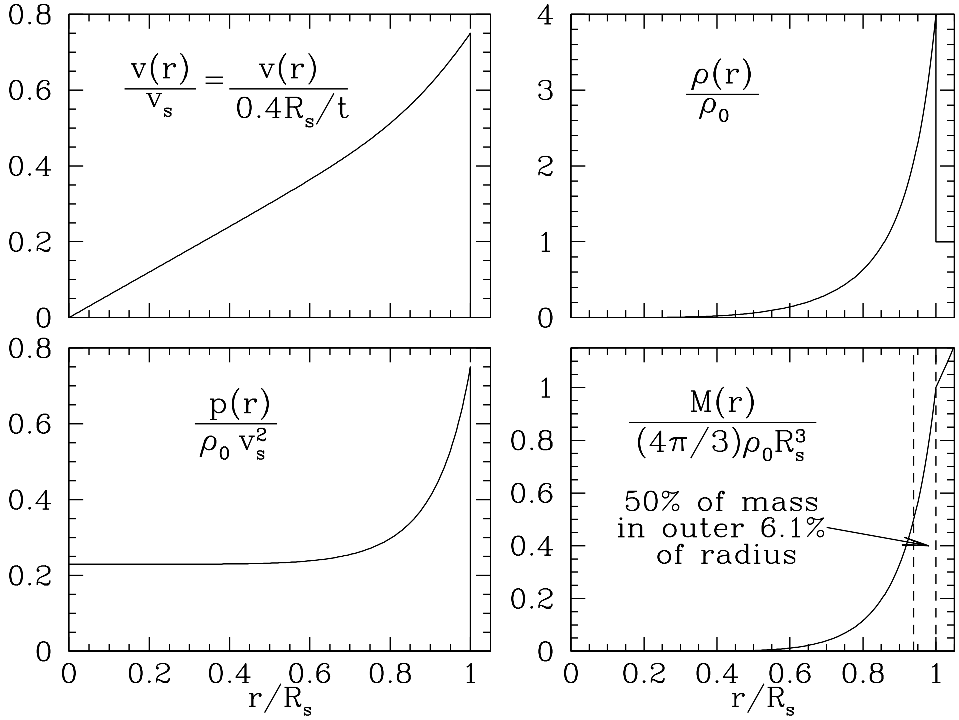


Figure 10: Sedov-Taylor expansion for $\gamma = 5/3$ [Draine 39.1].

erg s^{-1} cm^{-3}, this gives

$$\frac{dE}{dt} = -\int_0^{R_s} \Lambda \, 4\pi r^2 \, dr \tag{133}$$

Draine mentions a functional form for Λ that leads to:

$$t_{\rm rad} = 49.3 \times 10^3 \, {\rm yr} \, E_{51}^{0.22} n_0^{-0.55} \tag{134}$$

$$R_{\rm rad} = 7.32 \times 10^{19} \, {\rm cm} E_{51}^{0.29} \tag{135}$$

When $t \approx t_{\rm rad}$, the thermal pressure behind the shock has dropped significantly due to cooling. We call this the snowplow phase. There is now a dense shell of cool gas that is enclosing a hot central volume. The snowplow here refers to the fact that the dense shell mass is added to as the blastwave progresses outward.

The gas here is just cooling by adiabatic expansion. Adiabatic here just means that the gas does not transfer heat to its surroundings (via radiation). Hence this new phase is known as the radiative phase during which X-ray radiation becomes much less apparent and the remnant cools and disperses into the surrounding medium over the course of the next 10000 years.

At the beginning of the snowplow phase, Draine notes that the shock speed is ~ 150 km s^{-1}.

Draine gives the relevant expressions for the snowplow phase:

$$R_s \approx R_s(t_{\rm rad}) \left(\frac{t}{t_{\rm rad}}\right)^{2/7} \tag{136}$$

$$v_s \approx \frac{2}{7}\frac{R_s}{t_{\rm rad}} \left(\frac{t}{t_{\rm rad}}\right)^{-5/7} \tag{137}$$

4.4 Phase IV: Fadeaway

The shock speed declines with time until it becomes just an ordinary sound wave. Using our previous expressions, this occurs when

$$t_{\rm fade} \approx \left(\frac{(2/7)R_{\rm rad}/t_{\rm rad}}{c_s}\right)^{7/5} t_{\rm rad} \approx 1.87 \times 10^6 \text{ yr } E_{51}^{0.32} n_0^{-0.37} \left(\frac{c_s}{10 \text{ km s}^{-1}}\right)^{-7/5} \tag{138}$$

$$R_{\rm fade} \approx 2.07 \times 10^{20} \text{ cm } E_{51}^{0.32} n_0^{-0.37} \left(\frac{c_s}{10 \text{ km s}^{-1}}\right)^{-2/5} \tag{139}$$

Why does this happen? Internal pressure is not greater than external pressure. The SN could run into a dense structure, or radiative cooling may dominate.

4.5 Why would we care about this?

McKee & Ostriker (1977), in a classic paper, argued that blastwaves from SNe have a large impact on shaping the ISM. The envisioned an ISM consisting primarily of the CNM and the HIM. The WNM and WIM are restricted to the interface regions of the neutral clouds, and the WIM in direct contact with the HIM and photoionized by thermal emission from it. A blastwave propagates into these media.

The authors view the ISM as being composed of numerous small (spherical!) clouds of molecular gas, embedded in a diffuse hot ISM (HIM). Each cloud has an ionized halo (the WIM) maintained by the interstellar UV background. Between the ionized halo and the cloud itself, they suggest the presence of a neutral zone heated by interstellar X-rays.

It turns out this isn't really correct in detail, but nonetheless provides a useful framework.