ASTR 702
Stellar Stability (Chapter 6)

The energy output of stars on the main sequence is fairly constant (otherwise there wouldn't even be a main sequence!) and therefore we can expect that main sequence stars are stable. To test this, however, we want to take their equilibrium condition and slightly perturb it. This allows us to distinguish between an unstable and a stable equilibrium.

Stars have two types of equilibrium: thermal and hydrostatic. We'll discuss these in turn.

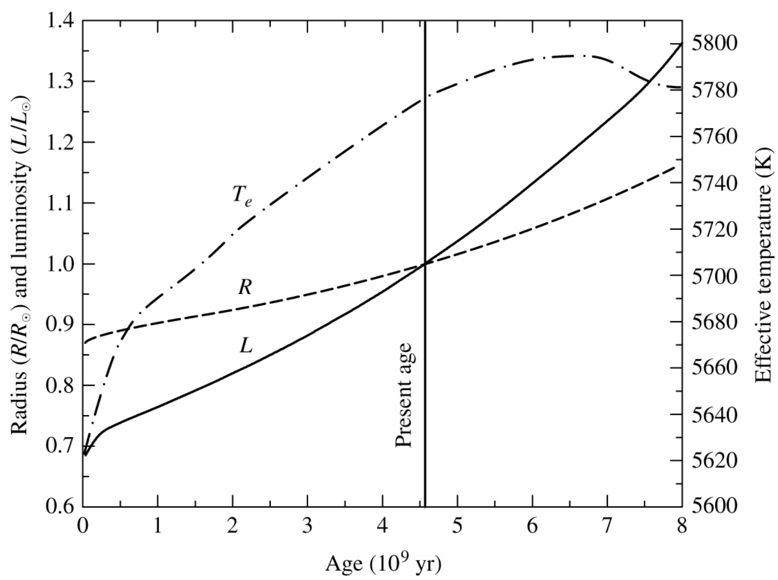

Figure 1: Evolution of the Sun on the Main Sequence.

Thermal Equilibrium

We can think through why main sequence stars are in thermal equilibrium. Let's imagine a main sequence star increases its fusion rate slightly. We know that this will provide an increase in outward pressure, so the star will expand. A larger star will have a lower core temperature, reducing the fusion rate. The reduced fusion rate will lead to less outward pressure, decreasing the size once again. This is the state of thermal equilibrium.

This argument relies on the positive link between temperature and heat production.

However, the argument breaks down for degenerate gases. In this case, a perturbation will still lead to expansion, but the temperature will remain the same and heat production will be the same. The increased luminosity will lead to a temperature increase, and more heat production, leading to a thermonuclear runaway. This could result in an explosion.

This isn't a death sentence for a degenerate star, however. It may expand to the point that it is no longer degenerate, and therefore the ideal gas law, which of course does have a temperature dependence, can lead to stability. This is the case in a nova.

We can derive these conditions mathematically. We had before that

$$P_c = (4\pi)^{1/3} B_n G M^{2/3} \rho_c^{4/3} \,. \tag{1}$$

and therefore

$$\frac{dP_c}{P_c} = \frac{4}{3}\frac{d\rho_c}{\rho_c} \,. \tag{2}$$

We can relate the pressure and density with an equation of state of the general form

$$P = \rho^a T^b \tag{3}$$

or

$$dP = (a\rho^{a-1}\, d\rho)\, T^b + (bT^{b-1}\, dT)\, \rho^a \tag{4}$$

and

$$\frac{dP}{P} = \frac{(a\rho^{a-1} d\rho) T^b + (bT^{b-1} dT)\rho^a}{\rho^a T^b} \tag{5}$$

$$\frac{dP_c}{P_c} = a\frac{d\rho_c}{\rho_c} + b\frac{dT_c}{T_c} \,. \tag{6}$$

And combining the relations we get

$$\left(\frac{4}{3} - a\right)\frac{d\rho_c}{\rho_c} = b\frac{dT_c}{T_c} \,. \tag{7}$$

We can see from our general form of the equation of state that $a = \gamma_a$ when considering degenerate gasses. For an ideal gas, $a = b = 1$.

So as long as $a < 4/3$, the sign of the density and temperature terms will be the same. In this case, contraction (ρ increases) leads to an increase in temperature. This is a stable equilibrium.

For degenerate gas, $a \gtrsim 4/3$ and $0 < b \ll 1$ and the density and temperature terms have opposite signs. Expansion (ρ decreases) actually increases the temperature! This is an unstable equilibrium. Similarly, degenerate stars should contract and cool (as for white dwarfs).

Thin shell instability

Assume we have a thin shell of mass Δm, temperature T, and density ρ with shell thickness $\ell = r - r_0$. If the shell has an increase in heat production, it will expand. The pressure in hydrostatic equilibrium is $P \propto r^4$, so

$$\frac{dP}{P} = -4\frac{dr}{r} \,. \tag{8}$$

The mass is $\Delta m = 4\pi r_0^2 \ell \rho$ so therefore

$$\frac{d\rho}{\rho} = -\frac{d\ell}{\ell} = -\frac{dr}{\ell} = -\frac{dr}{r}\frac{r}{\ell} \,. \tag{9}$$

We can substitute in for dr/r to find

$$\frac{dP}{P} = 4\frac{\ell}{r}\frac{d\rho}{\rho} \tag{10}$$

and we can use the general equation of state to find

$$\left(4\frac{\ell}{r} - a\right)\frac{d\rho}{\rho} = b\frac{dT}{T}\,. \tag{11}$$

Since b is positive,

$$4\frac{\ell}{r} > a \tag{12}$$

is the condition for stability. The main insight for this classic proof is that thin shells can be quite unstable.

Dynamical Equilibrium

In addition to thermal stability, stars must be dynamically stable. This condition refers to macroscopic gas motions, which hydrostatic equilibrium requires are nonexistent. We can test this stability criterion by perturbing the system, as before.

Let's start with hydrostatic equilibrium

$$\frac{dP}{dm} = \frac{Gm}{4\pi r^4} \tag{13}$$

so

$$P = \int_m^M \frac{Gmdm}{4\pi r^4} \tag{14}$$

and we had an expression for the density before:

$$\rho = \frac{1}{4\pi r^2}\frac{dm}{dr} \tag{15}$$

We can now perturb this system such that

$$r' = r - \epsilon r = r(1-\epsilon)\,. \tag{16}$$

If $\epsilon \ll 1$, we can expand it via the binomial expansion

$$(p \pm \epsilon)^n \approx 1 \pm n\epsilon \tag{17}$$

and therefore

$$\rho' = \frac{1}{4\pi r^2(1-\epsilon)^2}\frac{dm}{dr}\frac{dr}{dr'} = \frac{\rho}{(1-\epsilon)^3} \approx \rho(1+3\epsilon) \tag{18}$$

If we assume that the contraction is adiabatic,

$$\frac{P}{P'} = \left(\frac{\rho}{\rho'}\right)^\gamma \tag{19}$$

so

$$P'_{\mathrm{gas}} = P(1 + 3\epsilon)^{\gamma} \approx P(1 + 3\epsilon\gamma) \tag{20}$$

and by hydrostatic equilibrium

$$P'_{\mathrm{h}} = \int_{m}^{M} \frac{Gmdm}{4\pi r^4(1 - r\epsilon)^4} \approx P(1 + 4\epsilon) \tag{21}$$

After the perturbation, we don't expect P'_{gas} to be equal to P'_{h} anymore, and in fact $P'_{\mathrm{gas}} < P'_{\mathrm{h}}$. In order to restore equilibrium, $P'_{\mathrm{gas}} > P'_{\mathrm{h}}$. We thus require for stability that

$$P(1 + 3\gamma\epsilon) > P(1 + 4\epsilon)\,, \tag{22}$$

or $\gamma > 4/3$, as we had before for thermal stability.

Cases of dynamical instability

Because we require that $\gamma > 4/3$ for dynamical stability, we only need to ask when this is violated.

Relativistic-degenerate electron gas

The easiest condition conceptually is for relativistic degenerate electron gas, for which $\gamma = 4/3$. Remember that this is what leads to the Chandrasekhar mass, when the degeneracy pressure is no longer able to prevent collapse due to gravity.

Radiation pressure

The total energy of a star is $E = \Omega + U$, with $\Omega < 0$. If the star is stable, $E < 0$ (stronger gravity, larger Ω) or $|\Omega| > U$.

For radiation pressure, $\P/\rho = u_{\mathrm{rad}}/3$. Since $U = \int_0^M udm$, we can write

$$-\Omega = 3\int_0^M \frac{P}{\rho}dm = U_{\mathrm{rad}} \tag{23}$$

This star is unbound! So if radiation pressure dominates, we again have instability (this is the Eddington luminosity criterion).

Ionization-type processes

The final way that we can create dynamical instabilities is with processes in which the number of particles is not conserved, namely ionization.

In ionization, an atom loses an electron via a collision with another particle or a photon. The opposite process, recombination, is when an electron and an ion find each other and recombine. If the volume decreases and the density increases, the number of recombinations increases faster than the number of ionizations, and vice-versa for the volume increasing. Therefore, the number of particles changes in inverse proportion to the density.

Consider two systems, one with P, V, and N, where N is a constant under changes in volume, and one where N changes due to ionization. We know from the ideal gas law that $P \propto N/V$. Now let's compress these two systems from $V > V'$. In the first system, the pressure increases since $N/V' > N/V$. In the second system, since N' also increases, the pressure increases less. So the dependence of pressure on volume is weaker in the second system, which translates into a smaller value of γ, maybe less than 4/3. Therefore, in a cool stellar atmosphere, only completely ionized or completely neutral gas is dynamically stable.