Stellar
Stellar Stability

The energy output of stars on the main sequence is fairly constant (otherwise there wouldn't even be a main sequence!) and therefore we can expect that main sequence stars are stable. To test this, however, we want to take their equilibrium condition and slightly perturb it. This allows us to distinguish between an unstable and a stable equilibrium.

Stars have two types of equilibrium: thermal and hydrostatic (or dynamic). We'll discuss these in turn.
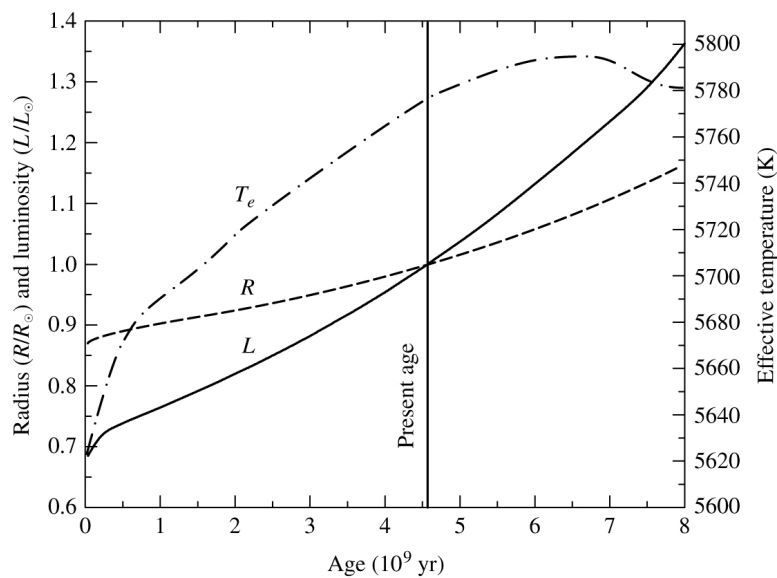

Figure 1: Evolution of the Sun on the Main Sequence.

Thermal Equilibrium

We can think through why main sequence stars are in thermal equilibrium. Let's imagine a main sequence star increases its fusion rate slightly. We know that this will provide an increase in outward pressure, so the star will expand. A larger star will have a lower core temperature, reducing the fusion rate. The reduced fusion rate will lead to less outward pressure, decreasing the size once again. This is the state of thermal equilibrium.

This argument relies on the positive link between temperature and heat production.

However, the argument breaks down for degenerate gases. In this case, a perturbation will still lead to expansion, but the temperature will remain the same and heat production will be the same. The increased luminosity will lead to a

temperature increase, and more heat production, leading to a thermonuclear runaway. This could result in an explosion.

This isn't a death sentence for a degenerate star, however. It may expand to the point that it is no longer degenerate, and therefore the ideal gas law, which of course does have a temperature dependence, can lead to stability. This is the case in a nova.

We can derive these conditions mathematically. We had before that

$$P_c = (4\pi)^{1/3} B_n G M^{2/3} \rho_c^{4/3} \, . \tag{1}$$

and therefore

$$\frac{dP_c}{P_c} = \frac{4}{3} \frac{d\rho_c}{\rho_c} \, . \tag{2}$$

We can relate the pressure and density with an equation of state of the general form

$$P = \rho^a T^b \tag{3}$$

or

$$dP = (a\rho^{a-1} \, d\rho) T^b + (bT^{b-1} \, dT) \rho^a \tag{4}$$

and

$$\frac{dP}{P} = \frac{(a\rho^{a-1} d\rho) T^b + (bT^{b-1} dT) \rho^a}{\rho^a T^b} \tag{5}$$

$$\frac{dP_c}{P_c} = a\frac{d\rho_c}{\rho_c} + b\frac{dT_c}{T_c} \, . \tag{6}$$

And combining the relations we get

$$\left(\frac{4}{3} - a \right) \frac{d\rho_c}{\rho_c} = b\frac{dT_c}{T_c} \, . \tag{7}$$

We can see from our general form of the equation of state that $a = \gamma_a$ when considering degenerate gasses. For an ideal gas, $a = b = 1$.

So as long as $a < 4/3$, the sign of the density and temperature terms will be the same. In this case, contraction (ρ increases) leads to an increase in temperature. This is a stable equilibrium.

For degenerate gas, $a \gtrsim 4/3$ and $0 < b \ll 1$ and the density and temperature terms have opposite signs. Expansion (ρ decreases) actually increases the temperature! This is an unstable equilibrium. Similarly, degenerate stars should contract and cool (as for white dwarfs).

Dynamical Equilibrium

In addition to thermal stability, stars must be dynamically stable. This condition refers to macroscopic gas motions, which hydrostatic equilibrium requires are nonexistent. We can test this stability criterion by perturbing the system, as before.

Let's start with hydrostatic equilibrium

$$\frac{dP}{dm} = \frac{Gm}{4\pi r^4} \tag{8}$$

so

$$P = \int_m^M \frac{Gm\,dm}{4\pi r^4} \tag{9}$$

and we had an expression for the density before:

$$\rho = \frac{1}{4\pi r^2}\frac{dm}{dr} \tag{10}$$

We can now perturb this system such that

$$r' = r - \epsilon r = r(1 - \epsilon). \tag{11}$$

If $\epsilon \ll 1$, we can expand it via the binomial expansion

$$(p \pm \epsilon)^n \approx 1 \pm n\epsilon \tag{12}$$

and therefore

$$\rho' = \frac{1}{4\pi r^2(1-\epsilon)^2}\frac{dm}{dr}\frac{dr}{dr'} = \frac{\rho}{(1-\epsilon)^3} \approx \rho(1 + 3\epsilon) \tag{13}$$

If we assume that the contraction is adiabatic,

$$\frac{P}{P'} = \left(\frac{\rho}{\rho'}\right)^\gamma \tag{14}$$

so

$$P'_{gas} = P(1 + 3\epsilon)^\gamma \approx P(1 + 3\epsilon\gamma) \tag{15}$$

and by hydrostatic equilibrium

$$P'_h = \int_m^M \frac{Gm\,dm}{4\pi r^4(1-\epsilon)^4} \approx P(1 + 4\epsilon) \tag{16}$$

After the perturbation, we don't expect P'_{gas} to be equal to P'_h anymore, and in fact $P'_{gas} < P'_h$. In order to restore equilibrium, $P'_{gas} > P'_h$. We thus require for stability that

$$P(1 + 3\gamma\epsilon) > P(1 + 4\epsilon), \tag{17}$$

or $\gamma > 4/3$, as we had before for thermal stability.

Cases of dynamical instability

Because we require that $\gamma > 4/3$ for dynamical stability, we only need to ask when this is violated.

Relativistic-degenerate electron gas

The easiest condition conceptually is for relativistic degenerate electron gas, for which $\gamma = 4/3$. Remember that this is what leads to the Chandrasekhar mass, when the degeneracy pressure is no longer able to prevent collapse due to gravity.

Radiation pressure

The total energy of a star is $E = \Omega + U$, with $\Omega < 0$. If the star is stable, $E < 0$ (stronger gravity, larger Ω) or $|\Omega| > U$.

For radiation pressure, $P/\rho = u_{rad}/3$. Since $U = \int_0^M u\,dm$, we can write

$$-\Omega = 3 \int_0^M \frac{P}{\rho}dm = U_{rad} \tag{18}$$

This star is unbound! So if radiation pressure dominates, we again have instability (this is the Eddington luminosity criterion).

Ionization-type processes

The final way that we can create dynamical instabilities is with processes in which the number of particles is not conserved, namely ionization.

In ionization, an atom loses an electron via a collision with another particle or a photon. The opposite process, recombination, is when an electron and an ion find each other and recombine. If the volume decreases and the density increases, the number of recombinations increases faster than the number of ionizations, and vice-versa for the volume increasing. Therefore, the number of particles changes in inverse proportion to the density.

Consider two systems, one with P, V, and N, where N is a constant under changes in volume, and one where N changes due to ionization. We know from the ideal gas law that $P \propto N/V$. Now let's compress these two systems from $V > V'$. In the first system, the pressure increases since $N/V' > N/V$. In the second system, since N' also increases, the pressure increases less. So the dependence of pressure on volume is weaker in the second system, which translates into a smaller value of γ, maybe less than 4/3. Therefore, in

a cool stellar atmosphere, only completely ionized or completely neutral gas is dynamically stable.

Convection

Convection refers to the bulk motion of gas. Hot gas is less dense (think of the ideal gas law) and therefore it will rise (for a star this means it will flow toward the surface of the star). Less dense gas will sink toward the center of the star. This process operates on parcels of gas, not individual atoms/ions. Convection may be regarded as a type of dynamical instability, although it does not have disruptive consequences. In fact, in spite of being a dynamic process, convection affects the structure of a star only as an effective heat carrier and as a mixing mechanism. All stars are convective, although to different degrees and in different regions.

Let's assume we have a mass element Δm within a star at point "1". We can characterize point 1 by a local mass density ρ_1 and pressure P_1, and these are the starting values within the mass element itself.

This mass element moves outward to point "2" that has density ρ_2 and pressure P_2. Since the pressure in a star decreases outward, $P_2 < P_1$, the surrounding pressure at point 2 will be lower than the pressure within the mass element. The element will therefore expand until the internal and external pressures are in balance. Because the dynamical timescale is much shorter than the thermal timescale, it is reasonable to assume that no heat exchange with the environment occurs while the mass element expands.

Hence the element undergoes an adiabatic change leading to a final density ρ_*, which is not necessarily equal to the density of its surroundings. If $\rho_* > \rho_2$, the mass element will descend back toward its initial position. We regard such a situation as stable, for any mass motion that may accidentally arise will be damped. Thus a stable convective cycle consists of parcels of gas rising, expanding, and falling. If, on the other hand, $\rho_* < \rho_2$, the element will continue its upward motion. In this case, the system is unstable against convection.

It is possible that a star be fully convective, all the way from the centre to the photosphere.

Let's look at Figure 6.3. The curves are "A" for adiabatic, S for one set of stellar parameters that results in an unstable condition and S' for another set that results in a stable condition. If the stellar configuration is described by S, then as the mass element ascends, $\rho_2 > \rho_*$, meaning instability, whereas if S' describes the stellar configuration, indicating stability against convection. In

conclusion, the condition for stability is

$$\left(\frac{dP}{d\rho}\right)_* < \left(\frac{dP}{d\rho}\right)_a \tag{19}$$

Multiplying both sides of this inequality by P/ρ gives

$$\frac{\rho}{P}\left(\frac{d\rho}{dP}\right) = \gamma_a \tag{20}$$

For an ideal gas, we have

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T}, \tag{21}$$

which gives us

$$\frac{P}{T}\left(\frac{dT}{dP}\right)_* < \frac{\gamma_a - 1}{\gamma_a} \tag{22}$$

If we multiply by dr/dr and rearrange, we find

$$\left|\frac{dT}{dr}\right|_* < \left(\frac{\gamma_a - 1}{\gamma_a}\right)\frac{T}{P}\left|\frac{dP}{dr}\right|_* \tag{23}$$

This is the upper limit for the magnitude of the temperature gradient.

Radiation vs. Convection

Stars have radiative zones and convective zones. Only low mass stars are fully convective. The Sun has a radiative zone in the interior and a convective zone outside of that. High mass stars have the opposite: radiative zones in their interiors and convective zones outside.

The Big Orange Book goes into a long derivation to determine when convection dominates over radiation. We can convince ourselves that by rearranging the above equation that convection dominates when

$$\frac{d\ln P}{d\ln T} < \frac{\gamma_a}{\gamma_a - 1}. \tag{24}$$

Although this equation is difficult to parse, we can say where convection will dominate: 1) in regions of high opacity where dT/dr would be too large for radiative transport ($dT/dr \propto \kappa$); this happens in the upper portion of the main sequence, where energy generation is due to the strongly temperature-dependent CNO cycle and convection is dominant in the core, 2) where ionization is occurring, which causes a large specific heat and low temperature gradient dT/dr the effective temperature decreases with decreasing mass,
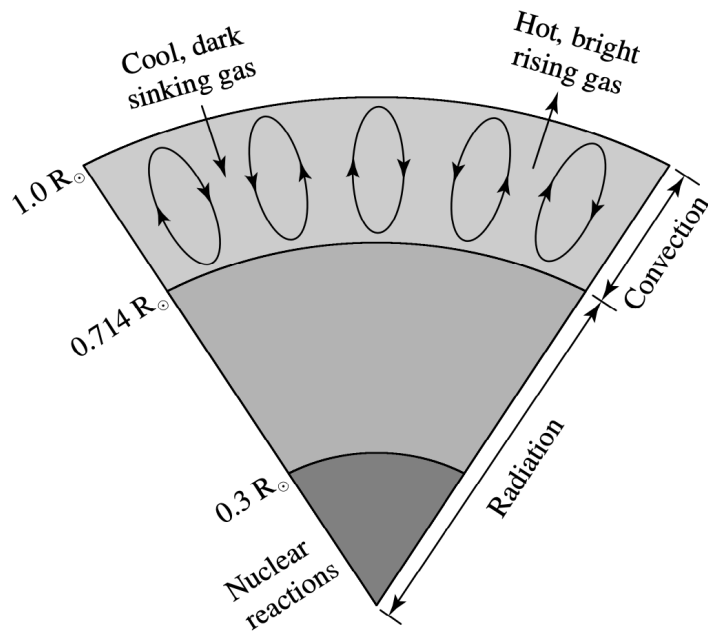
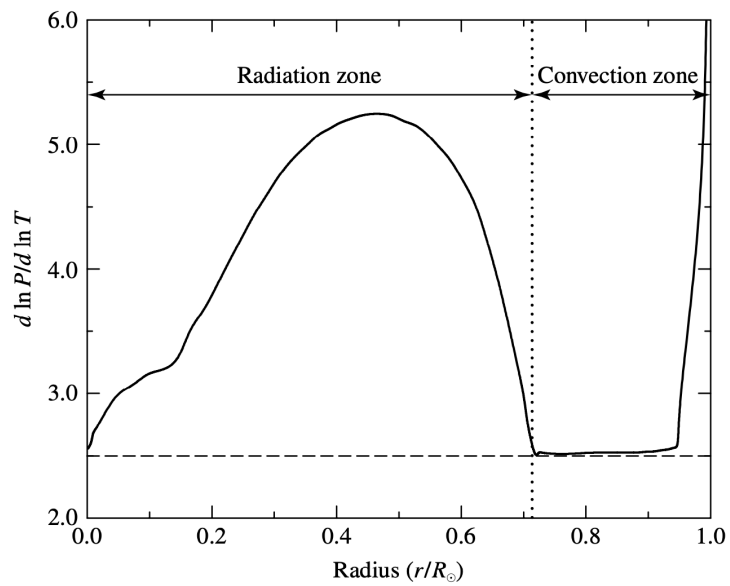Figure 2: Schematic of zones within the Sun.

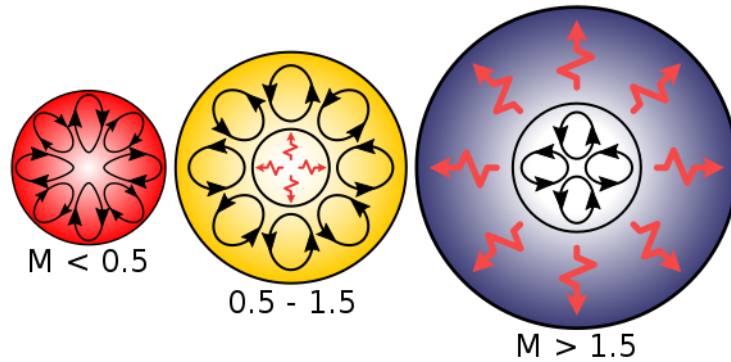Figure 3: Criterion for radiation/convection zones within the Sun.

Figure 4: Convection/Radiative zones by mass ranges.

the opacity increases, in part because of the location of the zone of hydrogen ionization, or 3) where the temperature dependence on nuclear energy generation is large, so *dF/dr* and *dT/dr* are large.

In summary, stability depends on the ability of the gas – particles and photons – at any given point to sustain the weight of the overlying layers by means of the pressure it exerts, so as to maintain exact balance despite possible perturbations. To this end, the pressure must depend strongly enough both on temperature and on density. Sensitivity to temperature is required in order to prevent thermonuclear runaways, and to density – so as to avoid collapse.