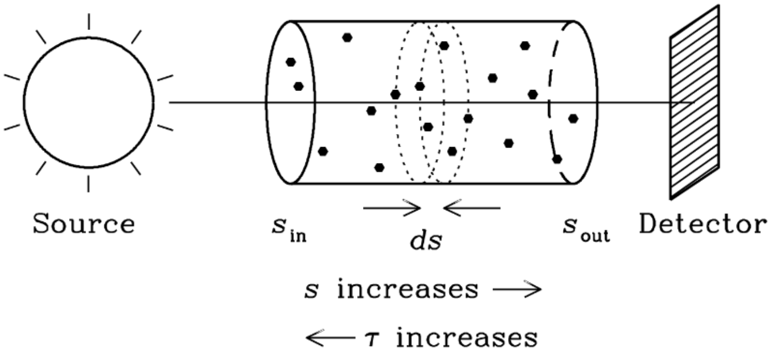
Stellar Radiation

Radiative transfer

Radiative transfer is the change in intensity dI_ν as radiation propagates from a source to the observer. Along the way, the emission will either be absorbed and scattered by intervening material, or it will encounter an emitting region. Consider an observation of a source shown schematically below (figure taken from "Essential Radio Astronomy") where the specific intensity is modified by absorption (attenuation) and/or emission from an intervening medium.



For attenuation, we can define a "linear absorption coefficient" κ_ν with units of cm^{-1}. *This is misleading since it contains contributions from both absorption and scattering!*[1] Note that this is not opacity or mass absorption coefficient, although both share the same notation and a similar definition! Sorry for the confusion. The amount of energy absorbed is proportional to the light intensity:

$$dI_\nu = -\kappa_\nu I_\nu ds, \tag{1}$$

where ds is the path. Absorption removes photons from the path, thus the negative sign. It is worth pointing out here that absorption excites atoms and molecules, and these atoms and molecules then re-emit. It this emission was beamed along ds there would be no change in intensity. Instead, the re-emitted light is more generally close to isotropic, so the emission is reduced.

For emission, we can define the emission coefficient j_ν as:

$$dI_\nu = j_\nu ds. \tag{2}$$

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[1] Absorption and scattering are easy to confuse. The main difference is that in scattering, the scattered radiation direction depends on the incident photon direction. Re-emission follows absorption, but this re-emission is isotropic.

Notice that there is no dependence on I_ν, in contrast to absorption. The units of j_ν are $\mathrm{erg\,cm^{-1}\,sr^{-1}\,s^{-1}}$.

The total change in intensity is therefore

$$dI_\nu = j_\nu ds - \kappa_\nu I_\nu ds, \tag{3}$$

or

$$\frac{dI_\nu}{ds} = j_\nu - \kappa_\nu I_\nu. \tag{4}$$

Equations 3 and 4 represent one form of the *Equation of Radiative Transfer*. This is one of the fundamental equations in astrophysics. All it is saying, however, is that the change in intensity along the path is just the emission (j_ν) minus the absorption ($\kappa_\nu I_\nu$).

Let's take the illustrative example of no emission. In this case

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu, \tag{5}$$

which has a solution

$$I_\nu(s) = I_{\nu,0} e^{-\kappa_\nu s}, \tag{6}$$

where $I_{\nu,0}$ is the unattenuated emission. The radiation intensity will decrease exponentially.

We can also define the dimensionless quantity of *optical depth* τ from

$$d\tau_\nu = -\kappa_\nu\, ds. \tag{7}$$

or

$$\tau_\nu = -\int \kappa_\nu\, ds, \tag{8}$$

where the integration is carried out over the path length. In most cases, we need only integrate over the source of interest. For example, if there is a gas cloud 20 kpc away that is 1 kpc thick, we may be able to only integrate over the 1 kpc of the cloud if the rest of the 20 kpc can be assumed to have no impact. For completeness,

$$I_\nu(\tau_\nu) = I_{\nu,0} e^{-\tau_\nu}, \tag{9}$$

The optical depth ranges from zero to infinity. Low values $\tau_\nu \ll 1$ are called "optically thin." These are things you can see through *at that particular frequency*. A good example is glass, which has a very low optical depth at optical frequencies, but actually has a high optical depth in the ultra-violet. High values $\tau_\nu \gg 1$ are called *optically thick*. A wall is optically thick at optical frequencies. A wall is optically thin at X-ray frequencies. Near $\tau \simeq 1$ we have to be careful - this is marginally optically thick.

If we rewrite things in terms of the optical depth, using $\frac{d\tau_\nu}{ds} = \kappa_\nu$,

$$\frac{dI_\nu}{d\tau_\nu} = \frac{j_\nu}{\kappa_\nu} - I_\nu. \tag{10}$$

We can further define the *Source function* S_ν

$$S_\nu = \frac{j_\nu}{\kappa_\nu} \tag{11}$$

Combining our expressions, we arrive a second form of the *Equation of Radiative Transfer*, this time using the optical depth and source function so that

$$\frac{dI_\nu}{d\tau_\nu} = S_\nu - I_\nu. \tag{12}$$

We will use this one from now on, because optical depth is a much better and more measurable parameter compared to actual linear depth.

In (full) thermodynamic equilibrium (TE) at temperature T, there is no change in intensity along the path and $\frac{dI_\nu}{d\tau} = 0$. In this case, $I_\nu = S_\nu = B_\nu(T)$, our old friend the Planck function. When is $I_\nu = B_\nu(T)$??? When $d\tau \rightarrow \infty$! Or in other words, when the optical depth is high, the intensity is that of a blackbody at temperature T. In this case, nothing else about the source matters, only its temperature.

This is a subtle, but extremely important point. For high optical depth sources, the only emission you can get out is that of a blackbody. You cannot for example get line emission. The source properties, aside from temperature, do not matter. The only thing you see is the *surface* emission. In fact, you only see down on average to the depth where the optical depth is unity. Think of a wall again, where you cannot determine how thick it is since you only see the paint layer (ok, so a wall actually is not a perfect blackbody since paint reflects light of different wavelengths....). Contrast this with glass. As glass get thicker, and thicker, we will notice more of a green hue. By determining *how* green it is, we can work out how thick it is. We will return to this point later.

Solutions to the Equation of Radiative Transfer

The deceptively simple equation of radiative transfer has had volumes written about its solutions. We can integrate the transfer function by multiplying by e^{τ_ν}. If we define $\tau_\nu = 0$ at $I_{\nu,0}$, we find

$$I_\nu(\tau_\nu) = I_{\nu,0}(\tau_\nu)e^{-\tau_\nu} + \int_0^{\tau_\nu} S_\nu(\tau')e^{-(\tau_\nu - \tau'_\nu)}\,d\tau' \tag{13}$$

The intensity I_ν at optical depth τ_ν is the initial (background) intensity $I_{\nu,0}$ attenuated by a factor $e^{-\tau_\nu}$, plus the emission $S_\nu d\tau^{'}$ integrated over the path, itself attenuated by the factor $e^{\tau_\nu - \tau^{'}_\nu}$. This final exponent represents "self-absorption." The material itself will absorb its own radiation. "Self-absorption" refers to absorption by one species (HI, CO, etc) by that species. If background radiation from e.g. HI is absorbed by optically thick HI, this is called self-absorption. This is known as the "formal solution to the equation of radiative transfer."

The difficulty in using Equation 13 is that in general we don't know how S varies with τ, because S depends on I, which is not known until S is known. It's a circular problem, which is why it is often solved computationally. It is worth examining this equation a bit more in limiting cases that allow us to simplify the integral:

$\boldsymbol{\tau}$ = 0

If the optical depth is zero, we get $I_\nu = I_{\nu,0}$, simply the background intensity back. If there is no optical depth, we get neither emission nor absorption (like a window!). This illustrates how emission and absorption are intimately related.

S constant

We can sometimes make the assumption that S_ν is a constant, so we can pull it out of the integral:

$$I_\nu(\tau_\nu) = I_{\nu,0}(\tau_\nu)e^{-\tau_\nu} + S_\nu \int_0^{\tau_\nu} e^{(-\tau-\tau')}\, d\tau' = I_{\nu,0}(\tau_\nu)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu}) \quad (14)$$

The first term on the right hand side is attenuation along the line of sight. The second one is emission along the line of sight.

S constant, LTE

In Local Thermodynamic Equilibrium, LTE, $S_\nu = B_\nu(T)$, so

$$I_\nu(\tau_\nu) = I_{\nu,0}(\tau_\nu)e^{-\tau_\nu} + B_\nu(1 - e^{-\tau_\nu}). \quad (15)$$

We will discuss LTE later, but essentially it means that for a small volume we can assume a single temperature that is also reflected in the level populations of the atoms and molecules.

S constant, LTE, Radio Regime

In the radio, we use the brightness temperature instead of the intensity. They are related by $I_\nu = \frac{2\nu^2}{c^2}kT_B$. We can also use the Rayleigh-Jeans approximation

$B_\nu(T) = \frac{2\nu^2}{c^2}kT$, with T here the kinetic temperature. Since these relationships both have the same constants, we can write

$$T_B = T_{B,0}e^{-\tau_\nu} + T(1-e^{-\tau_\nu}). \tag{16}$$

Note that the use of the Rayleigh-Jeans approximation here does not imply that the material is optically thick. It just implies that the emission is still modified by the optical depth.

S constant, LTE, Optically Thin

If $\tau \ll 1$, we get emission from the background radiation, as well as from along the line of sight. We can make the Taylor expansion substitution $e^{-\tau_\nu} \simeq 1-\tau_\nu$, so

$$I_\nu(\tau_\nu) = I_{\nu,0}(\tau_\nu)(1-\tau_\nu) + B_\nu\tau_\nu \simeq I_{\nu,0}(\tau_\nu) + B_\nu\tau_\nu. \tag{17}$$

The first term again is the background radiation attenuated by the ISM. The second term is the Planck function modified by the optical depth of the ISM. Notice that we can still have a blackbody-like spectrum even if it is optically thin, although it is modified by the optical depth (which is less than 1). In the case that $\tau_\nu = 0$, we of course only see the background radiation. In the radio regime,

$$T_B = T_{B,0}(1-\tau_\nu) + T\tau_\nu \simeq T_{B,0} + T\tau_\nu. \tag{18}$$

S constant, LTE, Optically Thick

If $\tau \gg 1$, $e^{-\tau_\nu} \to 0$, so

$$I_\nu = S_\nu. \tag{19}$$

If there is a blackbody in our line of sight, we don't see any emission from behind it. In radio astronomy, $T_B = T$ for optically thick emission, the kinetic temperature of the material (if in LTE).

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + B_\nu(T)(1-e^{-\tau_\nu}), \tag{20}$$

where $I_\nu(0)$ is the background radiation and τ_ν is the optical depth. So as $\tau \to \infty$, $I_\nu(\tau_\nu) \to 0 + B_\nu(T)(1-0) = B_\nu(T)$. To summarize, the Planck function has units of specific intensity or surface brightness, and in the limit of high optical depth, $I_\nu = B_\nu$.

There are a two important points about blackbody radiation.

1. The wavelength (or frequency) of peak intensity is inversely related to the temperature via Wien's Law:

$$\lambda_{\rm max} = \frac{0.2898}{T(\rm K)}\ \rm cm, \tag{21}$$

or

$$\nu_{\rm max} = 5.879 \times 10^{10} T(\rm K) \,. \tag{22}$$

We can derive these by setting the differential of B_λ or B_ν equal to zero. This tells us that hotter things peak at shorter wavelengths and higher frequencies.

The Sun for instance is 6000 K and peaks near 500 nm in the optical. A 10^4 K star peaks closer to 300 nm in the UV.

2. A hotter blackbody has a higher surface brightness intensity at *all* frequencies.

   It is important to remember that more intensity at all frequencies does not necessarily mean more energy! Think about burners on a stove. A small hot burner will have very intense radiation. A large cooler burner will have less intense radiation. But the larger one may boil water faster because although its intensity (surface brightness) is lower, it emits more total energy. What matters is the product of the surface brightness and the emitting area.

   Let's quantify this. To find the intensity (not the specific intensity), we integrate over all frequencies or wavelengths:

   $$B(T) = \int_0^\infty B_\nu(T) d\nu \,. \tag{23}$$

   After some math, this integral results in the expression

   $$B(T) = \frac{\sigma T^4}{\pi} \,, \tag{24}$$

   where σ is of course the Stephan-Boltzmann constant. In the case of an isotropic radiation field, which we can frequently assume, it can be shown that $F_\nu = \pi B_\nu$, so therefore $F = \sigma T^4$. This is of course the *Stephan-Boltzmann Law*. We are often interested in the total luminosity of an object (in erg $\rm s^{-1}$ or W):

   $$L = \int_S F dA \,, \tag{25}$$

   the flux integrated over the emitting surface. For spherical objects, this leads to $L = 4\pi r^2 \sigma T^4$, where r is the object's radius. Thus, the total energy output is related to the surface area and the temperature.

Stellar Radiation

A photon created deep in a stellar interior will have a hard time escaping to the surface. With each interaction, the photon direction is randomized. There is a pressure gradient from the inside out, and this can slowly direct the photons to the surface. Between interactions, a photon travels one mean free path ℓ. If it is to travel distance R, the radius of a star, C+O argue that it needs to have N interactions and

$$R = \ell\sqrt{N}\,. \qquad (26)$$

In terms of time, this is millions of years.

Although we often assume that we can see down to $\tau_\lambda = 1$, for stars it is actually closer to $\tau_\lambda = 2/3$ (see C+O, Section 9.4). For a sightline passing through the edge of the star, $\tau_\lambda = 2/3$ is only reached at the outer layers, which are cooler. For a sightline looking straight at the star, we can see down further into the interior. This effect is known as “limb darkening.”

Kirchoff’s Laws

We can divide a spectrum into **“line”** and **“continuum”** components. A spectral line is a decrement or excess intensity on top of the continuum [draw figure of spectral line]. Spectral lines are produced from discrete (quantized) transitions, most simply by electrons through electric dipole radiation, although many more emission mechanisms are possible. Continuous radiation can come from a variety of different emission mechanisms. The distinction is whether the energy levels are quantized (emission and absorption lines) or not (continuous).

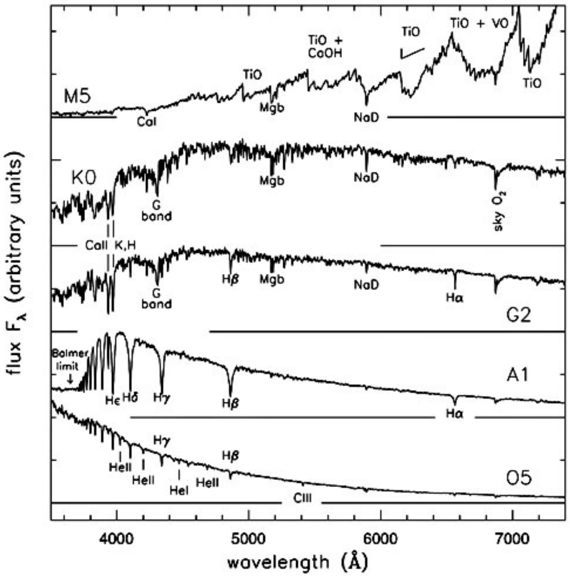

Figure 1: Spectra for stars of various spectral types, showing absorption lines on top of the continuum.

There are two broad types of line spectra:

Absorption lines (decreases in intensity relative to nearby continuum) are produced when a cold gas is between a continuous spectrum source and the detector. Atoms in the cold gas absorb photons from the continuous spectrum source. The photons are then emitted in random directions, removing intensity along the line of sight. What is a continuous spectrum source? Stars and quasars are pretty close (especially quasars). Most stellar spectra are absorption line spectra, however, because the outer layers of the photosphere are less dense and colder than the core.

Emission lines at discrete wavelengths are produced when the detector sees photons emitted directly from a (hot) gas. The wavelengths of the emission lines are due to electronic transitions within the atoms and are therefore unique to each atom.

The highest intensity radiation at a given frequency is at high optical depth ($\tau_\nu \rightarrow \infty$), which is given by the Planck function. Let's imagine that you have continuous radiation of low optical depth (a "gray-body"), an emission line's peak intensity is limited to the Planck function intensity.

The spectrum you observe depends on the density (the optical depth) of the object, and the viewing direction. Observing the same object from a different direction will give you a different signal. **Kirchoff's Laws** tell us how to interpret the spectra we observe. There are multiple sets of Kirchoff's Laws, so it is safe to assume that Kirchoff was wicked smart and interesting at parties.

Kirchoff's three laws of spectra are:

- A dense object produces light with a continuous (blackbody) spectrum. Kirchhoff also coined the term blackbody radiation because he was a show-off. You emit blackbody radiation, with a peak in the infrared.
- A hot diffuse gas produces an emission line spectrum due to electronic transitions within the gas. Fluorescent lights are a good example.
- A hot dense object surrounded by a cool tenuous gas (i.e., cooler than the hot object) produces an absorption line spectrum. The absorption lines are at exactly the same wavelengths as the emission lines for a given element, and are also due to electronic transitions.

Line Broadening

Emission and absorption lines are not delta functions, they are broadened by some mechamisms. What can broaden spectral lines?
1) Thermal doppler motion due to gas particles at a given temperature, "Doppler broadening." Not all particle speeds will be the same of course. As the temperature increases, the range of speeds does too.

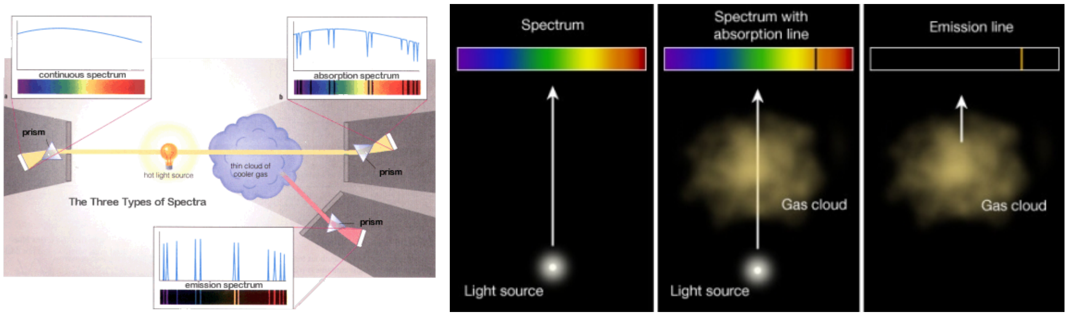

Figure 2: Kirchoff's Laws

2) "Turbulent broadening" due to the fact that there are bulk motions within any gas.
3) Natural broadening due to the fact that the energy "level" is not a single value, and
4) Pressure or collisional broadening, which changes the energy levels.

Each of these mechanisms has an associated line shape. Doppler broadening is Gaussian, and turbulent is usually assumed to be Gaussian as well. Natural and Collisional broadening are "Lorenzian," which is like a Gaussian but with much larger "wings." All four processes operate at the same time, resulting in a "Voigt" profile with a Gaussian core and Lorentzian wings. Since the Lorentzian wings are at low intensity, usually something best approximated by a Gaussian is observed.

Gaussians are magical functions. A normalized Gaussian takes the form of

$$\phi(\nu) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{(\nu-\nu_0)^2}{2\sigma^2}\right)}., \tag{27}$$

where ν_0 is usually the line center and σ is the one-dimensional velocity dispersion. You can see that the line will be of maximum intensity when $\nu = \nu_0$ at line center, then $\phi(\nu_0) = \frac{1}{\sigma\sqrt{\pi}}$.

We can also define the full-width at half-maximum (FWHM) as

$$FWHM = \sqrt{8\ln 2}\,\sigma = 2.355\sigma\,. \tag{28}$$

For thermal gas with particles of mass m following a MB distribution,

$$\sigma = \left(\frac{kT}{m}\right)^{1/2} = 9.12\left(\frac{T_4}{m/\mathrm{amu}}\right)^{1/2}\ \mathrm{km\,s^{-1}}\,. \tag{29}$$

The constant goes to 21.47 $\mathrm{km\,s^{-1}}$ for the FWHM.

The thermal line width increases with increasing temperature and decreases with increasing mass.

Gaussians have the amazing property that the area under the curve is approximately the FWHM times the peak. They are also their own Fourrier transform pair, and a Gaussian convolved with a Gaussian leads to another Gaussian. This last point is very important in astronomy. Frequently, the spectral response of your instrument can be assumed to be Gaussian, and the source or spectral line Gaussian as well. Therefore, you will observe a Gaussian.

Why do we get Gaussians? We had the MB velocity distribution before:

$$f(v) = \left(\frac{m}{2\pi kT}\right)^{3/2} 4\pi v^2 e^{-mv^2/2kT} \,. \tag{30}$$

What you have probably never seen is that in 1D the $4\pi v^2$ term goes away. This term arises in 3D due to the density of velocity states available (see Hyperphysics site). This is a Gaussian! Looking at Equation 27, the original exponent numerator was $-(\nu - \nu_0)^2$ and is now $-v^2$. The original exponent denominator was $2\sigma^2$ and is now $2kT/m$. We can define:

$$\sigma_v = \left(\frac{kT}{m}\right)^{1/2} = 9.12\left(\frac{T_4}{m/\mathrm{amu}}\right)^{1/2} \mathrm{km/s}\,, \tag{31}$$

where T_4 is the temperature in units of 10^4 K. The FWHM is 2.355σ, or

$$FWHM = 2.355\sigma = 21.47\left(\frac{T_4}{m/\mathrm{amu}}\right)^{1/2} \mathrm{km/s}\,. \tag{32}$$

This leads to:

$$f(v) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma_v} e^{-u^2/2\sigma_v^2} \,. \tag{33}$$

You can see how the velocity dispersion σ_v goes into the Gaussian.

We can assume that turbulence adds another Gaussian term, and we can add the thermal and turbulent components in quadrature, which leads to

$$\sigma = \left(\frac{kT}{m} + v_{\mathrm{turb}}^2\right)^{1/2} \,, \tag{34}$$

We often don't know v_{turb}, but given a spectral line we can determine it from the linewidth if we can estimate the temperature.

The natural width arises due to the uncertainty principle: $\Delta E \Delta t \sim \hbar$. Here, $\Delta t = A_{\mathrm{u}\ell}^{-1}$, so short-lived states have large uncertainties in energy. Because for electric dipole radiation $A_{\mathrm{u}\ell} \propto \nu^3$, $\Delta E \propto \nu^3$ and natural broadening is important at high frequencies. It gives rise to a Lorentzian profile function

$$\phi(\nu) = \frac{\gamma}{4\pi^2}\frac{1}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2} \,, \tag{35}$$

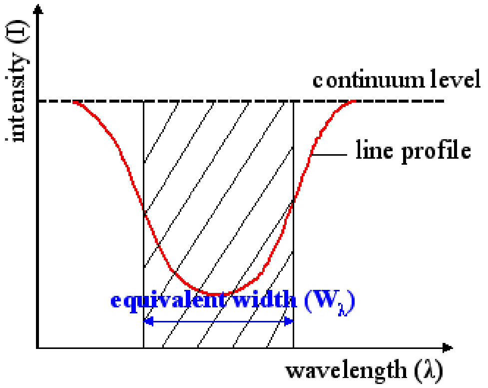

Figure 3: Equivalent width.

where γ is a constant for each species related to the spontaneous decay rates

$$\gamma = \Sigma A_{\mathrm{u}\ell}. \tag{36}$$

Like natural broadening, collisional broadening distorts the energy levels, leading to an additional Lorentzian term that can be combined with natural broadening:

$$\phi(\nu) = \frac{\Gamma}{4\pi^2} \frac{1}{(\nu - \nu_0) + (\Gamma/4\pi)^2}, \tag{37}$$

where $\Gamma = \gamma + 3\nu_{\mathrm{col}}$, and ν_{col} is the collision frequency, $\nu_{\mathrm{col}} = n\sigma v$.

In the radio, Doppler and turbulent broadening are typically the largest. At high frequencies (X-ray), natural braodening can make a large contribution.

Characterizing Spectral Lines

It is often preferable to fit a line profile to a spectral line in order to characterize its emission. An alternative method, called the "equivalent width," is insensitive to the exact profile. The equivalent width is the width, in wavelength or frequency units of a rectangular area equal to that of the spectral line:

$$W_\lambda = \int (1 - F_\lambda/F_0) d\lambda \tag{38}$$

The intensity of a spectral line changes with the optical depth at line center. This is usually parameterized as the change in the optical depth, or the change in "column density," the integral of the number density over the pathlength.

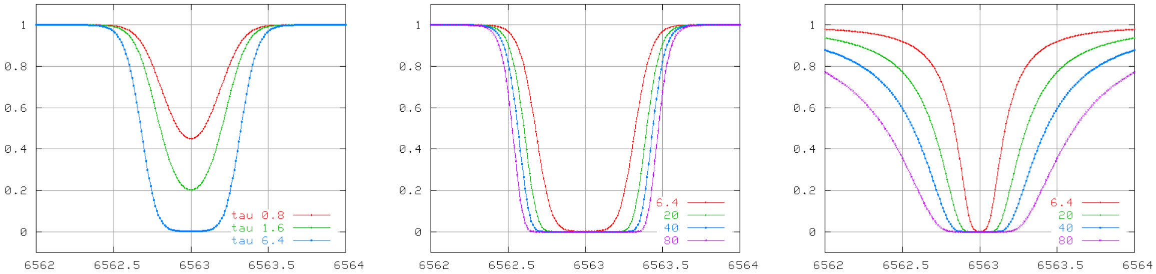

Figure 4: Line saturation for various optical depths (http://spiff.rit.edu/classes/phys440/lectures/curve/curve.html).

Let's assume we have an absorption line and no source function:

$$\frac{I_\nu}{I_{\nu,0}} = e^{-\tau_\nu} \tag{39}$$

Initially, the more absorbing atoms there are, the stronger the absorption line. If we keep adding absorbers to the path, however, eventually the line will saturate. A saturated line means that at line center (the most probable speed in a MB distribution) no more intenisity can be added. After it saturates, the equivalent width grows slowly, because there are few atoms with the requisite speeds, until the growth of the line wings becomes important.

The column density is a fundamental quantity in astronomy, since it is directly related to what we measure. Since $\kappa_\nu \propto n$ and $\tau_\nu = \int \kappa_\nu ds$, $\tau \propto \int n\, ds = N$. This is the number of particles along a $1\,\mathrm{cm}^2$ cylinder path.

This leads to three important regimes:
1) The "Linear" regime ($\tau \lesssim 5$) where the equivalent width W is proportional to the column density, $W \propto N$.
2) The "saturated" regime where $W \propto \sqrt{\ln N}$, and
3) The "damping" regime where collisional broadening takes over, $W \propto \sqrt{N}$.

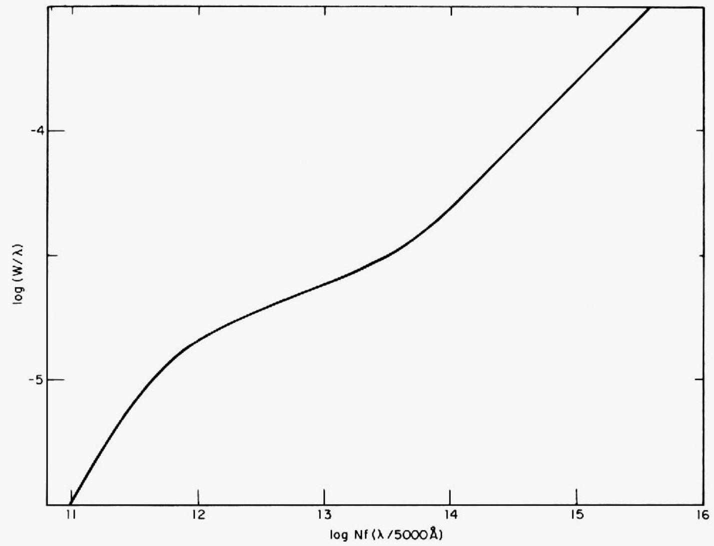

Figure 9.22 A general curve of growth for the Sun. (Figure from Aller, *Atoms, Stars, and Nebulae*, Revised Edition, Harvard University Press, Cambridge, MA, 1971.)

Figure 5: The curve of growth. The x-axis is parameterized in terms of the oscillator strength, which is a parameter unique for each transition.