Stellar - Stellar Evolution
Prialnik Ch 12, C+O Chapter 12

Star Formation

Stars have to form somehow, because stars exist. The formation of stars must involve collapse of a molecular cloud. The larger molecular clouds, called "giant molecular clouds," have mean densities of $\sim 10^2$ cm^{-3} or so, and sizes of maybe 100 pc ($\sim 10^{18}$ m). A higher-density region within them must collapse down to $\sim 10^9$ m (the size of a star), with densities of $\sim 10^{24}$ cm^{-3} (the mean density of a star).

When does this collapse occur? When gravity overcomes pressure. The condition where gravity and pressure are in balance is of course called "hydrostatic equilibrium." One treatment says that a cloud not in hydrostatic equilibrium that will collapse has a characteristic size of the "Jeans radius" and mass of the "Jeans Mass" (the condition of instability is the "Jeans Instability"). We will derive these quantities first from the hydrostatic equilibrium condition. We will also do the same derivation using the Virial theorem.

Jeans Mass from Hydrostatic Equilibrium [Following Wikipedia page]

The Jeans mass is named after the British physicist Sir James Jeans, who considered the process of gravitational collapse within a gaseous cloud. He was able to show that, under appropriate conditions, a cloud, or part of one, would become unstable and begin to collapse when it lacked sufficient gaseous pressure support to balance the force of gravity. The cloud is stable for sufficiently small masses (at a given temperature and radius), but once this critical mass is exceeded, it will begin a process of runaway contraction until some other force can impede the collapse (fusion, in the case of a star). He derived a formula for calculating this critical mass as a function of its density and temperature. The greater the mass of the cloud, the smaller its size, and the colder its temperature, the less stable it will be against gravitational collapse.

Hydrostatic equilibrium is:

$$\frac{dP}{dr} = -\frac{G\rho(r)M_r}{r^2}, \tag{1}$$

where M_r is the enclosed mass, P is the pressure, $\rho(r)$ is the density of the gas at r, G is the gravitational constant and r is the radius. The equilibrium is stable if small perturbations are damped and unstable if they are amplified. In general, the cloud is unstable if it is either very massive at a given temperature or very cool at a given mass for gravity to overcome the gas pressure.

Let's say we have a spherical molecular cloud cloud of radius R, mass M, and sound speed c_s. Compression of this region can only proceed at approximately the sound speed, which gives a characteristic time of:

$$t_{\rm sound} = \frac{R}{c_s} \tag{2}$$

for sound waves to cross the region. Gravity will attempt to contract the system even further, and will do so on a free-fall time,

$$t_{\rm ff} = \sqrt{\frac{3\pi}{32G\rho}}. \tag{3}$$

We have collapse when $t_{\rm ff} < t_{\rm sound}$. In this case, the collapse is fast enough that the cloud cannot re-establish equilibrium, which takes place over the timescale given by the sound speed.

It is worth taking a slight detour here to describe how long these free fall times are. For large scales, the growth time for the Jeans instability is

$$\tau_J \simeq 2.3 \times 10^4 {\rm yr} \left(\frac{10^6 \ {\rm cm}^{-3}}{n_H} \right)^{1/2} \tag{4}$$

For $n_H = 1000 \ {\rm cm}^{-3}$, this is about 0.7 Myr. Free fall time (collapse timescale for a pressure-less gas) is:

$$\tau_{ff} = \left(\frac{3\pi}{32G\rho_0} \right)^{1/2} = 4.4 \times 10^4 \ {\rm yr} \left(\frac{10^6 \ {\rm cm}^{-3}}{n_H} \right)^{1/2} \tag{5}$$

For $n_H = 1000 \ {\rm cm}^{-3}$ this is 1.4 Myr - slightly longer than growth time.

OK, back to the Jeans mass and radius. The resultant Jeans radius R_J is therefore:

$$\lambda_J \simeq \frac{c_s}{\sqrt{G\rho}} \tag{6}$$

The speed of sound is

$$c_s = \sqrt{\frac{\gamma P}{\rho}}, \tag{7}$$

where γ is the adiabatic index, which is 7/5 for molecular gas and 5/3 for monotonic gas. The pressure $P = nkT = \rho/\mu kT$ assuming an ideal gas, with mean mass μ, so we have

$$R_J \simeq \sqrt{\frac{kT}{G\mu\rho}}. \tag{8}$$

The real definition gives a factor of order unity out front:

$$R_J \simeq \sqrt{\frac{15kT}{4\pi G\mu\rho}} \simeq (0.4\ \mathrm{pc})\left(\frac{c_s}{0.2\ \mathrm{km\ s^{-1}}}\right)\left(\frac{n}{10^3\ \mathrm{cm^{-3}}}\right)^{-1/2}. \qquad (9)$$

All scales larger than the Jeans length are unstable to gravitational collapse, whereas smaller scales are stable.

Perhaps the easiest way to conceptualize Jeans Length is in terms of a close approximation, in which we rephrase ρ as M/r^3. The formula for Jeans' Length then becomes:

$$R_J \approx \sqrt{\frac{k_B T r^3}{G M \mu}}, \qquad (10)$$

and therefore $R_J = r$ when $kT = \frac{GM\mu}{r}$. In other words, the cloud's radius is the Jeans Length when thermal energy per particle equals gravitational work per particle. At this critical length the cloud neither expands nor contracts. It is only when thermal energy is not equal to gravitational work that the cloud either expands and cools or contracts and warms, a process that continues until equilibrium is reached.

We can recast this in terms of the "Jeans mass":

$$M_J = \left(\frac{4\pi}{3}\right)\rho R_J^3 = \left(\frac{5kT}{G\mu}\right)^{3/2}\left(\frac{3}{4\pi\rho}\right)^{1/2} \simeq (2\ \mathrm{M_\odot})\left(\frac{c_s}{0.2\ \mathrm{km\ s^{-1}}}\right)^3\left(\frac{n}{10^3\ \mathrm{cm^{-3}}}\right)^{-1/2}. \qquad (11)$$

The Jeans mass M_J is just the mass contained in a sphere of radius R_J. It is useful to remember that $M_J \propto T^{3/2}\rho^{-1/2}$. Thus, stars can form most efficiently (when mass is low) in low temperature, high density locations where the Jeans mass is not as great.

The above is an illustrative and wrong derivation! Jeans assumed that the collapsing region of the cloud was surrounded by an infinite, static medium. The pressure in hydrostatic equilibrium is therefore less than that required, and the mass is therefore too high. We will fix this problem below.

A larger issue is that because all scales greater than the Jeans length are also unstable to collapse, any initially static medium surrounding a collapsing region will in fact also be collapsing. As a result, the growth rate of the gravitational instability relative to the density of the collapsing background is slower than that predicted by Jeans' original analysis. This flaw has come to be known as the "Jeans swindle".

The Jeans Mass from the Virial Theorem

The Virial Theorem says that for *gravitationally bound* systems in equilibrium, the total energy is one-half of the time-averaged potential energy. The gravitationally bound aspect is important. If the system is not gravitationally bound, the Virial Theorem will not hold. Such cases give rise to contraction or expansion.

We can also derive the Jeans mass using the Virial theorem. Like the condition of hydrostatic equilibrium, the Virial theorem describes a system in equilibrium. If the kinetic energy of a system is U and the gravitational potential energy is Ω, the simplest incarnation of the Virial theorem says that $2U + \Omega = 0$. An expanding gas cloud will have more kinetic energy than gravitational ($2U > -\Omega$) and a contracting cloud will have more gravity ($2U < -\Omega$).

Each particle in a gas cloud has kinetic energy, $E = 3/2kT$, so the total kinetic energy $U = \Sigma_i^N E_i = 3/2NkT$, where N is the number of particles. Easy!

The gravitational potential energy in spherical shell of mass dm is

$$d\Omega = -G\frac{mdm}{r}\,. \tag{12}$$

From mass conservation, we know

$$dm = 4\pi r^2 \rho dr\,. \tag{13}$$

Putting these two expressions together, we find

$$d\Omega = Gm4\pi r\rho dr\,. \tag{14}$$

We can integrate this from 0 to R to get the gravitational potential, but not if we don't know how ρ and M_r depend on r. For a constant density sphere (polytropic index $n = 0$),

$$\rho \simeq \bar{\rho} = \frac{M}{4/3\pi R^3} \tag{15}$$

so

$$M_r \simeq 4/3\pi r^3 \bar{\rho}\,. \tag{16}$$

The integral then gives

$$\Omega \simeq -\frac{3}{5}\frac{GM^2}{R}\,, \tag{17}$$

which is the gravitational potential for an "isothermal" sphere. We had this before, but it's so fundamental that it's worth re-deriving.

Therefore,

$$3NkT = -\frac{3}{5}\frac{GM^2}{R}\,. \tag{18}$$

We can replace $N = M/\mu$, with μ the mass per particle, and $R = (3M/4\pi\rho)^{1/3}$ to get

$$M_J = \left(\frac{5kT}{G\mu}\right)^{3/2} \left(\frac{3}{4\pi\rho}\right)^{1/2} \tag{19}$$

The same as before! Physics!

Fragmentation

Of course this is a simplification – a single cloud does not collapse down to $r = 0$. What happens to complicate the collapse? As the cloud collapse, density rises. Since the collapse is isothermal, a rising density means the Jeans mass of the cloud is falling, so small pieces of the cloud start to collapse on their own. A rising density also means a declining free fall time, so these small dense clumps collapse faster than the overall cloud.

Instead of one giant cloud undergoing a monolithic collapse, the cloud fragments into small collapsing pieces. So what stops this fragmentation? As the density rises, the opacity rises. At some point during the collapse and fragmentation process, the opacity rises high enough that the energy created during the collapse is absorbed within the star itself – it begins to heat up. Since the energy is not lost from the cloud, we call this an adiabatic collapse. Higher temperature means higher pressures (the ideal gas law), which halt the free collapse of the star. Since the cloud absorbs all the gravitational energy of collapse, it heats up, and it starts to act like a blackbody.

At what mass does this happen? We can balance the rate of energy loss through gravitational collapse to the rate at which the cloud radiates blackbody energy, and, solving for the mass, we find $M \approx M_\odot$. In other words, collapse halts when the fragment masses reach star-like masses.

Bonner-Ebert Spheres

In a more realistic scenario, the density is centrally peaked. In this case, the gravitational energy is

$$U = -\frac{3}{5}\alpha\frac{GM^2}{R}, \tag{20}$$

where $\alpha > 1$ for centrally peaked density profiles. Mouschovias & Spitzer (1976) find $\alpha \approx 1.67$ for numerical models of clouds on the verge of collapse.

In our above consideration of the Virial theorem, we neglected external pressure and magnetic energy. If we consider the former, with the above modification to the gravitational potential, we arrive at the “Bonner-Ebert mass”

(Bonner 1956; Ebert 1957):

$$M_{\rm BE}(p_0) = \frac{225}{32\sqrt{5}\pi}\frac{c_s^4}{(aG)^{3/2}}\frac{1}{\sqrt{p_0}} = 0.26\left(\frac{T}{10\ \rm K}\right)^2\left(\frac{10^6\ \rm cm^{-3}\ K}{p_0/k}\right)^{1/2} M_\odot \qquad (21)$$

Remember how we said that the Jeans mass neglected some rather important things? Well, the Bonner-Ebert mass is basically the same as the Jeans mass, $M_{\rm BE} \approx 1.18 M_J$. The 18% change is due to the fact that the cloud itself affects the hydrostatic equilibrium assumption before.

Given the typical temperature and pressures of molecular clouds, the Bonner-Ebert mass is about a Solar mass, so it is probably no surprise that this is approximately the peak of the IMF.

Of course we still neglect the magnetic fields. The magnetic energies are similar to the kinetic energies, and so can contribute to the pressure. But the physics of magnetic fields is less understood, and more complicated, so we'll omit discussion!

Formation of Actual Stars

During the formation of stars, cores more or less free-fall collapse. The free fall time depends inversely on the density, so the central part collapses first, then the outer parts. What would provide resistance? [Pressure!] What would pressure be unimportant? [Cooling from molecular lines!]. This "cooling" of course releases energy that we can detect. This energy peaks in the sub-millimeter to far-infrared. How much energy is liberated? Where does it go?

The collapse is not uniform. Angular momentum must be preserved, so as the size scale of the cloud decreases $\sim$7 orders of magnitude, it must spin 7 times faster! It seems that most molecular clouds are very slowly rotating, providing initial angular momentum.

As it collapses, an accretion disk forms that provides material to the growing star. As the star accretes mattter, jets form perpendicular to the disk. The system is now known as a "T Tauri" star, after its namesake.

Pre-Main Sequence Evolution

With a protostar still getting its energy from gravitational collapse, the opacity in the outer layers increases, due to the H^- ion. This causes the protostar to become convective and to lose luminosity. This phase is known as a "Hayashi track."

We can derive the location of the Hayashi tracks on the H-R diagram. Let's assume our usual polytrope

$$P = K\rho^{\gamma} = K\rho^{1+1/n} \tag{22}$$

We had a definition of K in the derivation of the Lane-Emden equation:

$$K^n = C_n G^n M^{n-1} R^{3-n}\,, \tag{23}$$

where $C_n = \frac{4\pi}{(n+1)^2}\frac{R_n^{n-3}}{M_n^{n-1}}$. We can also say

$$L = 4\pi R^2 \sigma T_{\rm eff}^4\,, \tag{24}$$

where $T_{\rm eff}$ is the effective temperature of the photosphere. We need an equation that relates the pressure. We have hydrostatic equilibrium, which gives us

$$P_R = \frac{GM}{R^2}\int_R^{\infty} \rho dr\,, \tag{25}$$

where the integration is from R to ∞, where the pressure vanishes. We can recast the RHS using the definition of opacity, which can be modeled as (at R)

$$\kappa = \kappa_0 \rho_R^a T_{\rm eff}^b \tag{26}$$

and since at the photosphere, $\tau = 1 = \int \kappa\rho ds$,

$$\kappa_0 \rho_R^a T_{\rm eff}^b \int_R^{\infty} \rho dr = 1 \tag{27}$$

and therefore

$$P_R = \frac{GM}{R^2\kappa_0}\rho_R^{-1} T_{\rm eff}^{-b}\,. \tag{28}$$

Whew! This gives us a set of four equations, which we can solve to get a relationship between luminosity, mass, and temperature. Our book does so to get

$$\log L = A \log T_{\rm eff} + B \log M + {\rm constant}\,, \tag{29}$$

with A and B being constants that depend on the polytropic index and a and b from before. The book notes that the signs of A and B are opposite.

The Hayashi tracks are the pre-main sequence paths a star makes. Thus A and B tell us how this star looks in an H-R diagram. At low mass, A is large and B is small, and the Hayashi tracks are almost vertical. At higher mass A is smaller and B is more negative, leading to diagonal lines in the H-R diagram.

Eventually, the protostars begin fusion, and are officially stars. EXCEPT for "brown dwarfs," which never reach high enough temperatures for fusion. Stars

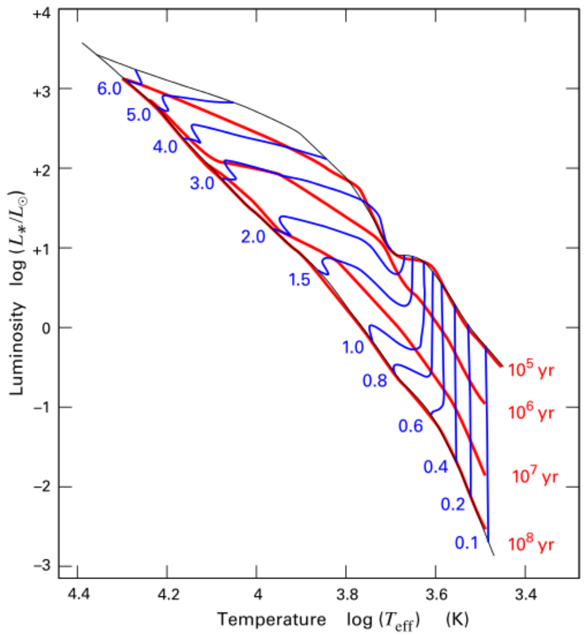


Figure 1: Pre-main sequence evolution. The red lines are not the evolution of individual stars, but rather show discrete ages. High mass stars evolve more quickly onto the main sequence.

then move to higher temperature at about the same luminosity. This limit is at a mass of 0.072 $M_\odot$. Together with the Eddington limit, this sets the mass range that stars can have.

Once stars begin fusion, they are on the main sequence. When they first reach the main sequence, they are on what is called the "zero-age main sequence." Stars continue to evolve a bit on the main sequence, and the zero-age main sequence is their starting point.

The Initial Mass Function (revisited)

Massive stars are rare, and low mass stars are common. The stellar birth rate is given by the "initial mass function," or IMF.

We can see from the IMF that the most common mass is around 0.5 $M_\odot$ and the numbers decrease at lower and higher masses (only for logarithmic bins! For linear, the distributions are $\sim$flat below $\sim$ 0.5 $M_\odot$. Note that this is the *initial* mass! To get the present-day mass function, we have to account for stellar lifetimes, and as we'll see, low-mass stars live for a long time.

We can define the number of stars formed at a given time within a given volume with masses from M to $M + dM$ as

$$dN = \Phi(M)dM \tag{30}$$

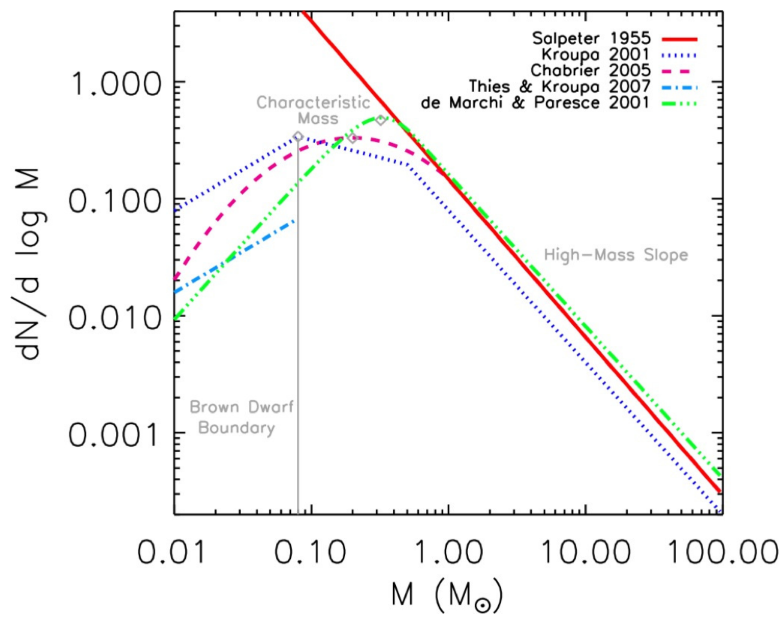


Figure 2: The initial mass function as derived by various authors. Note that the binning is logarithmic - for linear bins, there is no peak in the distributions.

The birth function $\Phi(M)$ can be determined empirically. At high masses,

$$\Phi(M) \propto M^{-\alpha}. \tag{31}$$

We can also define the initial mass function $\xi(M)$ as the amount of mass locked up in stars with masses from M to $M + dM$:

$$\xi(M) = M\frac{dN}{dM} \tag{32}$$

and therefore

$$\xi(M) \propto \left(\frac{M}{M_\odot}\right)^{-\alpha+1} \tag{33}$$

Like many things stellar, α depends on the mass. At the high-mass end, we find agreement between authors that $\alpha = 2.35$. This was first derived by Salpeter (1955) and the initial mass function of this form sometimes bears his name.

We can use the IMF to compute the amount of mass returned to the ISM. Let ζ be the mass initially locked up in shars in the range $M_{\rm min}$ to $M_{\rm max}$

$$\zeta = \int_{M_{\rm min}}^{M_{\rm max}} MdN \int_{M_{\rm min}}^{M_{\rm max}} \xi(M)dM. \tag{34}$$

The fraction returned, R will depend on the mass range. Massive stars have $R = 1$. Stars with $M < 8\,M_\odot$ have $R = (R - R_{\rm WD})/R$. And stars with $M < 0.7\,M_\odot$

have $R = 0$ because they have not lived long enough. The mass return is therefore

$$\eta = \int_{M_{\min}}^{M_{\max}} \xi(M)R(M)dM \quad (35)$$

The fractional mass returned is η/ξ, which our book finds is about 1/3.

We can also estimate the number of stars of a given type if we know the maximum mass of such stars.

$$N_{\mathrm{MS}} = \int_{M_{\min}}^{M\mathrm{tp}} dN = \int_{M_{\min}}^{M\mathrm{max}} \Phi(M)dM \,. \quad (36)$$

Main Sequence Evolution

Stars spend 80-90% of their lifetimes on the main sequence where they happily convert hydrogen into helium.

Our book, Chapter 7, deals with stellar evolution during the main sequence hydrogen burning phase by discussing zones on the $\log T, \log \rho$ plane. This is an interesting way to examine stellar evolution. All processes in stars have characteristic temperature and density ranges.

First, we can define zones based on the equation of state. We have three main equations of state, for ideal gas $P = K_0 \rho T$, for electron degenerate gas $P = K_1 \rho^{5/3}$, and for relativistic electron degenerate gas $P = K_2 \rho^{4/3}$. We also have radiation dominated stars with $P = 1/3aT^4$.

By equating these, we can arrive at dividing lines in the $\log T, \log \rho$ plane. To separate relativistic degenerate from degenerate gas,

$$K_1 \rho^{5/3} = K_2 \rho^{4/3} \quad (37)$$

so

$$\rho = \left(\frac{K_2}{K_1}\right)^3 . \quad (38)$$

Since these are constants, it's a horizontal line in the $\log T, \log \rho$ plane. For degenerate and ideal gas

$$K_1 \rho^{5/3} = K_0 \rho T \quad (39)$$

So this is a diagonal line.

$$\rho = \left(\frac{K_0}{K_1}\right)^{3/2} T^{3/2} \quad (40)$$

Finally, between radiation and ideal

$$K_0 \rho T = 1/3aT^4 \quad (41)$$

gives us another diagonal line

$$\rho = 1/3aK_0^{-1}T^3 \tag{42}$$

Zones of nuclear burning

We had before that

$$q = q_0\rho^m T^n \,. \tag{43}$$

For most nuclear burning processes $m = 1$ (triple alpha being the main exception) and $m >> 1$. Thus, in the $\log T, \log\rho$ plane, nuclear burning is mostly vertical lines.

Evolution of a Main Sequence Star

Eqns of Stellar Structure

We know from before that the central pressure:

$$P_c = C\frac{M^2}{R^4} \tag{44}$$

The ideal gas law implies that

$$T_c = C\frac{\mu m_H}{k\rho}\frac{M^2}{R^4} \tag{45}$$

If we let $R = \left(\frac{3M}{4\pi\bar{\rho}}\right)^{1/3}$ then:

$$T_c = C\left(\frac{4\pi}{3}\right)^{4/3}\frac{\mu m_H}{k}M^{2/3}\bar{\rho}^{1/3} \tag{46}$$

From radiative diffusion:

$$L = -\frac{64\pi\sigma r^2}{\kappa\rho}T^3\frac{dT}{dr} \tag{47}$$

Approximating $dT/dr \sim T/R$, then:

$$L \propto \frac{RT^4}{\kappa\rho} \tag{48}$$

If we adopt the Kramers opacity $\kappa = \kappa_0 \rho T^{-3.5}$:

$$L \propto \frac{M^{5.53} \rho^{0.166} \mu^{7.5}}{\kappa_0} \tag{49}$$

Since M is constant, and there is a low dependence on ρ (which is probably also close to constant in time):

$$\frac{L(t)}{L(0)} = \left[\frac{\mu(t)}{\mu(0)}\right]^{7.5} \tag{50}$$

The result is that as Hydrogen is converted into Helium, the Sun becomes more luminous. Early on, it was 25% less luminous.

log T log ρ

We can be a little more rigorous if we take the polytropic models from before. In Chapter 5, we had

$$P_c = (4\pi)^{1/3} B_n G M^{2/3} \rho_c^{4/3} \,, \tag{51}$$

where the "c" subscripts denote central quantities. If we use the ideal gas law, we can determine the relationship between density, mass, and temperature:

$$\rho_c = \frac{K_0^3}{4\pi B_n^3 G^3} \frac{T_c^3}{M^2} \,. \tag{52}$$

For a star of a given mass, the central density varies as the central temperature cubed. In the $\log T, \log \rho$ plane, this relation is a straight line of slope 3. *Therefore, stars cannot move along arbitrary paths in the* $\log T, \log \rho$ *plane.*

If the star becomes relativistic, there is no temperature dependence because the equation of state has changed:

$$\rho_c = 4\pi \left(\frac{B_n G}{K_1}\right)^3 M^2 \,, \tag{53}$$

which is a horizontal line in the $\log T, \log \rho$ plane.

We can define the main sequence as following a straight line in the H-R diagram:

$$\log L = \alpha \log T_{\text{eff}} + \text{constant} \tag{54}$$

Our book does some algebra of the main equations of stellar structure to arrive at

$$L^{1 - \frac{2(n-1)}{3(n+3)}} \propto T_{\text{eff}}^4 \,, \tag{55}$$

where n is the temperature dependence index for nuclear fusion (not the polytropic index). Therefore,

$$\log L = \frac{12(n+3)}{3(n+3)-2(n-1)} \log T_{\rm eff} + {\rm constant} = \frac{12(n+3)}{n+11} \log T_{\rm eff} + {\rm constant}. \quad (56)$$

For the proton-proton chain with $n=4$, $\alpha=5.6$. For the CNO cycle with $n=16$, $\alpha=8.4$. This accounts for the changing slope of the main sequence.

As we saw when discussing the Sun, stars are fairly stable over their main sequence lifetimes in terms of their luminosities, radii, and temperatures. Stars do change their energy output slightly throughout their lives (their temperatures change less). Thus, they move upward on the H-R diagram throughout their lives. The physical mechanism is that as stars deplete their H in favor of He, the mean particle mass μ increases. Thus gravity becomes stronger, the core contracts, fusion increases, and the overall radius (which is set by the balance between thermal pressure and gravitational) increases. Stars convert hydrogen into helium in their cores. This process increases the mean particle mass. From the ideal gas law,

$$P = \frac{\rho k T}{\mu m_H}, \quad (57)$$

we see that increases in mass per particle must be compensated by the temperature or the density going up. Both of these will necessarily increase the reaction rates, leading to an increase in luminosity. The strong dependence of the P-P chain on the energy output increases the luminosity for small changes in temperature.

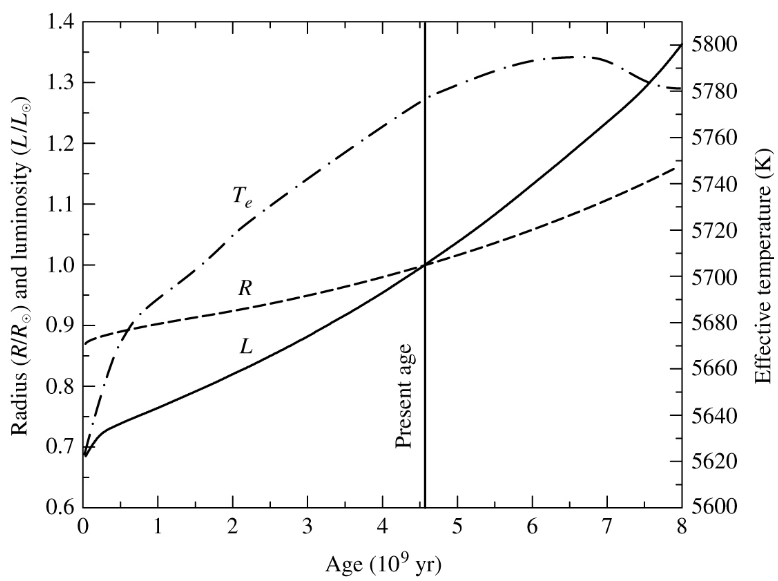


Figure 3: Evolution of the Sun on the Main Sequence.

Post-main sequence evolution

This evolution has some basic principles that we already know well:

- Stars lose mass, and therefore fuel, via fusion
- This loss of fuel reduces the central pressure
- reduced pressure causes the star to contract
- A contracted star may have other fusion processes available.
- These other fusion processes may lead to more central pressure, increasing the luminosity and size of the star.

Additionally, more massive elements tend to “sink” toward the center, increasing the metallicity there.

The evolution of stars differs dramatically for stars of different masses. The dividing mass between two different evolutionary sequences is about 8 $M_{\odot}$. All post-main-sequence evolution begins when stars run out of hydrogen *in their cores*. A star can have hydrogen outside the core that is not available for fusion (because the temperature is too low) and this same evolutionary sequence will begin.

Low-mass ($M < 8\ M_{\odot}$) Main Sequence Evolution

Eventually, stars run out of hydrogen in their cores. Stars still are plenty hot enough for fusion, but without hydrogen, they have trouble doing fusion in their cores. For instance, the triple-alpha process requires temperatures unreachable by stars on the main sequence, and without hydrogen, both P-P and CNO are not possible. Thus the core shrinks, giving off energy transformed from gravitational potential. The luminosity increases.

Now imagine that the central core of the star is depleted in Hydrogen, and that fusion is taking place primarily in a shell surrounding the central core. What happens to the central core?

Let us consider the virial theorem for a shell. Now, we have a virial theorem similar to that which we considered for clouds and cores with an internal and external pressure (see Lecture 4 equations, Section 2).

$$\int_0^{V_c} P\,dV = P_s V_c + \frac{1}{3}\alpha\frac{GM_c^2}{R_c} \tag{58}$$

Now assume an isothermal gas where $P = \rho kT/\mu m_H$. Dividing by the volume, we get

$$P_s = \frac{3}{4\pi}\frac{kT}{\mu m_H}\frac{M_c}{R_c^3} - \frac{\alpha G}{4\pi}\frac{M_c^2}{R_c^4} \tag{59}$$

If we set $P_s = 0$, we get a minimum radius:

$$R_0 = \alpha\frac{\mu m_H}{3k}\frac{M_c\mu_c}{T_c} \tag{60}$$

Below this radius, there is no stable configuration. However, if there is external pressure, we can find a maximum value of P_s, by finding out where $dP_s/dR_c = 0$. At this point:

$$0 = -\frac{9}{4\pi}\frac{kT}{\mu m_H}\frac{M_c}{R_c^4} + \frac{\alpha G}{\pi}\frac{M_c^2}{R_c^5} \tag{61}$$

Giving the solution:

$$R_1 = \frac{4\alpha G}{9R_\star}\frac{M_c\mu_c}{T_c} \tag{62}$$

Thus, if $R < R_0$, the core is unstable if the pressure was 0. If there is an external pressure, the core is unstable if $R < R_1$. It is stable for $R > R_1$. At R_1, there is a maximum in the pressure where if the pressure is exceeded, the core will collapse. By substituting this back into the equation for P_s, we get

$$P_{s,\max}(M_c) = C_1\frac{T_c^4}{M_c^4\mu_c^4} \tag{63}$$

The surrounding pressure is due to the weight of the envelope. We can estimate it using the equation of hydrostatic equilibrium (assuming $M_c < M_\star$):

$$\frac{dP}{dm} = \int_0^{M_\star}\frac{Gm}{4\pi r^4}\,dm \tag{64}$$

Since $r < R$, where R is the radius of the star, we get the inequality:

$$P_{\text{env}} > \int_0^{M_\star}\frac{Gm}{4\pi R^4}\,dm = \frac{GM_\star^2}{8\pi R_\star^4} \tag{65}$$

Thus, the core is unstable if

$$P_{s,\max}(M_c) = C_1\frac{T_c^4}{M_c^2\mu_c^4} \geq \frac{GM_\star^2}{8\pi R_\star^4} \tag{66}$$

Now, we can use the equation

$$T_c = C_2 \frac{\mu_{\rm env} m_H}{k} \frac{GM_\star}{R_\star} \tag{67}$$

giving the final relationship

$$\frac{M_c}{M_\star} \leq C_3 \left(\frac{\mu_{\rm env}}{\mu_c} \right)^2 \tag{68}$$

Schönberg and Chandrasekhar derived that $C_3 = 0.37$.

What are $\mu_{\rm env}$ and μ_c? Let us assume that the envelope is dominated by Hydrogen, while the core is dominated by Helium. The total value of μ is given by:

$$\frac{1}{\mu} = \frac{1}{\mu_{e^-}} + \frac{1}{\mu_{\rm ion}} \approx \frac{3+5X}{4} \tag{69}$$

This gives us $\mu_{\rm env} = 0.6$ and $\mu_{\rm core} = 1.33$.

The critical ratio is when the mass of the Helium approaches the size:

$$\frac{M_c}{M_\star} \leq 0.37 \left(\frac{0.6}{1.33} \right)^2 \sim 0.1 \tag{70}$$

Thus, when 10% of the Hydrogen is converted into Helium, the star becomes unstable. At this point, it enters the Red Giant phase.

This situation has a problem: the core is no longer producing radiation and therefore a significant source of pressure is removed. There is still thermal pressure, but no radiation pressure. If we are to prevent the collapse of the core, we need an additional source of pressure - degeneracy!

Evolution off the Main Sequence

After core contraction, stars begin fusion in a shell around their cores. This reaction actually produces more energy than the stars did on the main sequence lives, moving them upward on the H-R diagram. Much of this energy does not make it out of the star, increasing its radius and lowering its temperature. Stars therefore move to the right in the H-R diagram. The stellar cores are made up of the byproducts of fusion, the “ash.” This material sinks toward the core.

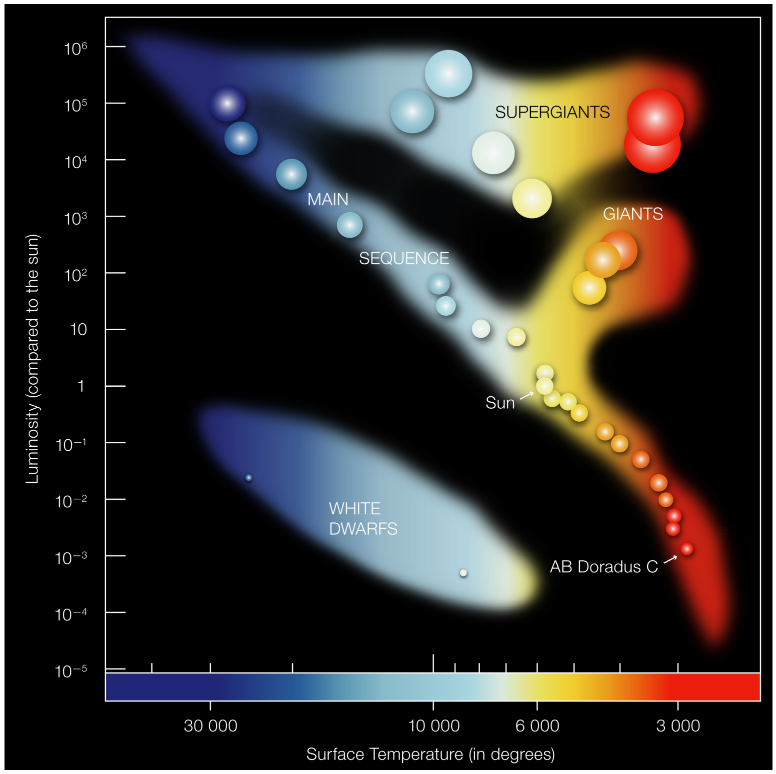


Figure 4: H-R diagram

Subgiant Branch

Eventually the core contracts enough that the outer layers of the core at at a temperature high enough to sustain fusion. This phase is known as the "subgiant branch." During the subgiant branch, the luminosity is higher than it was on the main sequence. H shell burning produces more energy than H core burning. Throughout the subgiant branch, the star's temperature decreases and its radius increases. These effects nearly offset so the luminosity is stable.

Red Giant Branch

With the decrease in temperature, there are more H^- ions formed in stars' photospheres. This leads to a high opacity and a further lowering of the temperature. As a result, convection takes over as the dominant energy transport method ($d \ln P / d \ln T \lesssim 2.5$). This convective zone starts at the surface and reaches deep into the stellar interiors, down to the hydrogen burning shell. The energy transport is so efficient that the luminosity skyrockets. This is known as the "red giant branch."

The transport of materials is known as the "first dredge up."

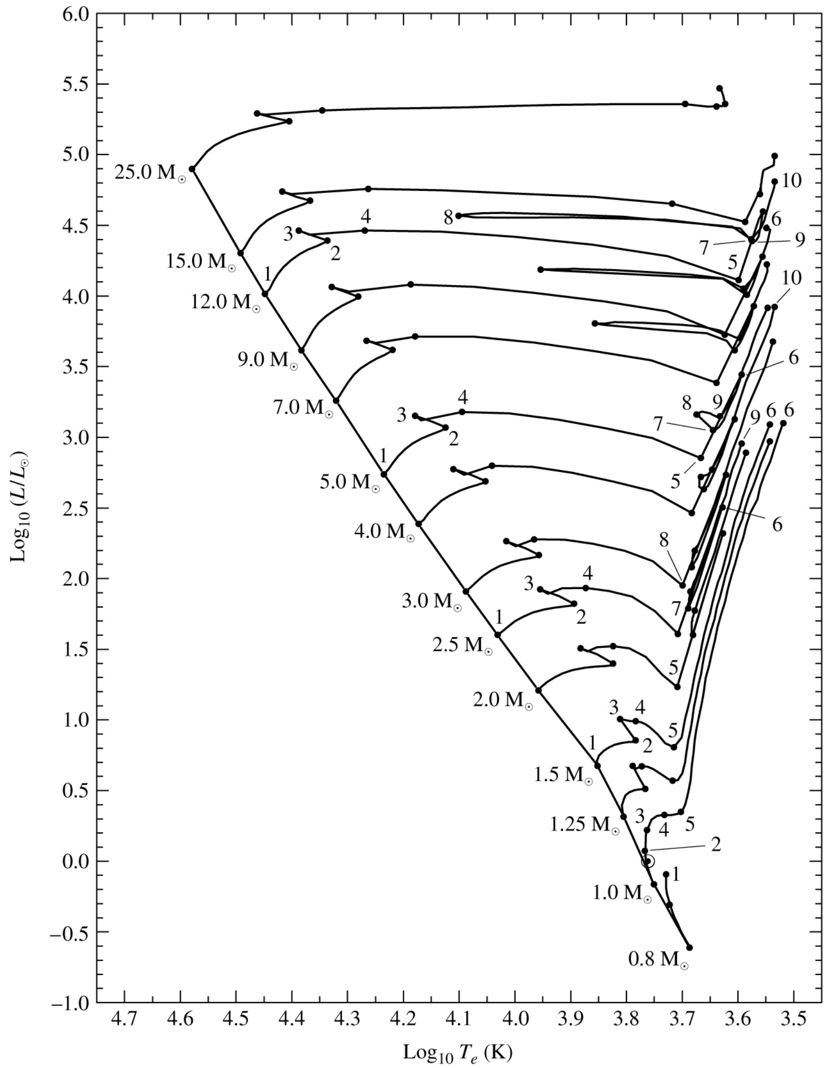


Figure 5: Main sequence and post-main sequence evolution of stars.

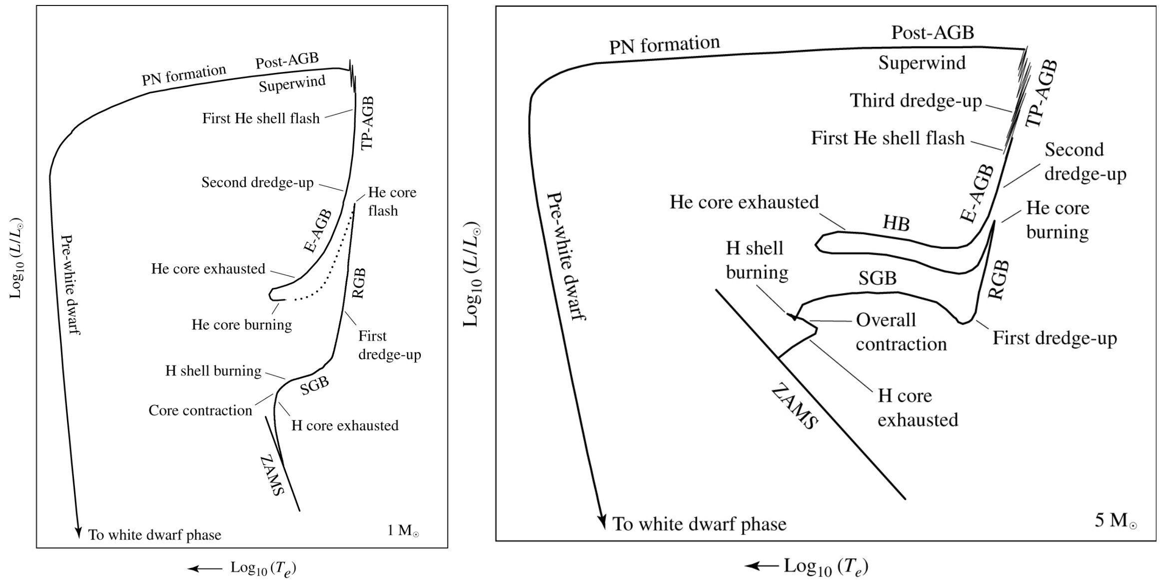


Figure 6: Evolution off the main sequence for 1 (left) and 5 $M_\odot$ (right) stars.

Red Giant Tip

Eventually the central temperature and pressure are high enough to fuse helium nuclei through the triple-alpha process. Some C is fused with He to make O at this stage.

Initially, most of the energy still comes from H shell fusion, but the triple alpha process quickly expands the core due to its extreme energy output. This cools the H shell to the point where it cannot create significant energy. Removed of its main fusion source, the star decreases in luminosity.

Helium Flash

Stars with masses less than 1.8 $M_\odot$ create electron-degenerate cores. Surprisingly, their cores are at lower temperatures than the material outside at higher radius. This happens because the neutrinos from the core carry away significant energy.

Eventually, the core reaches a temperature and pressure great enough for the triple-alpha process. This releases tons of energy all at once, equivalent to the luminosity of an entire galaxy. The "helium flash" only lasts a few seconds though.

Evolution beyond the helium flash is difficult to predict, and so stellar models sometimes momentarily stop here.

Stars with masses greater than 1.8 $M_\odot$ do not have a helium flash.

The Horizontal Branch

The core at this point contracts, increasing the energy output. The temperature also increases, leading to evolution blue-ward. This is called the “horizontal branch.”

Eventually, all the He in the core has been converted into C and O. The star at this point contracts and reddens.

The contracting core leads to a dual-shell burning phase on the horizontal branch, with nested shells of hydrogen and helium burning.

Early Asymptotic Giant Branch

The “Early Asymptotic Branch” is the dual shell burning phase, although the He shell is doing all the energy output. At this point the luminosity rises and the temperature decreases, analogous to what happens in the H shell burning phase.

Once again, convection takes over in the outer layers, leading to a “second dredge up.”

Thermal-Pulse Asymptotic Giant Branch

Now the hydrogen burning shell has reignited. Yay! But the helium shell is running out of fuel. The hydrogen shell is dumping its helium ash onto the helium shell. This results in intermittent explosions of He fusion, leading to pulsations. The He runs out again and the process repeats, leading to pulsations.

Due to the fact that their atmospheres are tenuously bound, AGB stars lose significant fractions of their mass, up to $10^{-4}\ M_\odot$ per year. AGB stars are a significant source of dust production in the Universe.

One class of such AGB stars are known as long-period variables (LPVs). LPVs have pulsation periods of 100 to 700 days.

Post-AGB Evolution

The evolution of stars post-AGB is heavily dependent on the initial mass. Stars with $M < 8\ M_\odot$ will go on to form planetary nebulae and white dwarfs. Stars with $M >\ M_\odot$ will go supernova (more on this later).

The extreme mass loss exposes the stellar core, moving the stars leftward on the H-R diagram. Pulsations contribute to the mass loss as more mass is lost at each pulsation.

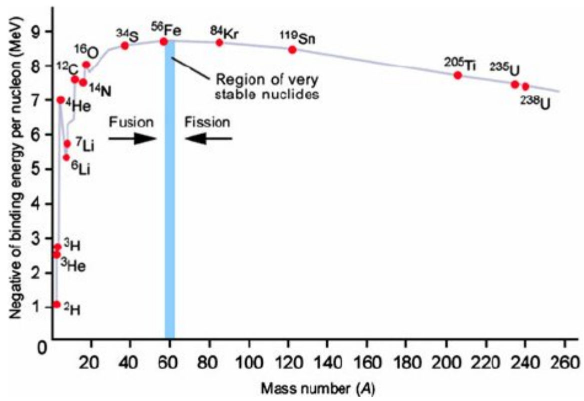


Figure 7: The binding energy curve.

Eventually, the core is completely exposed. This core is composed of carbon and oxygen, and is known as a “white dwarf.”

Planetary Nebulae

The term “planetary nebula” is a misnomer, because they have nothing to do with planets. Instead, they are the layers of material expelled during AGB pulsations that remain around white dwarfs.

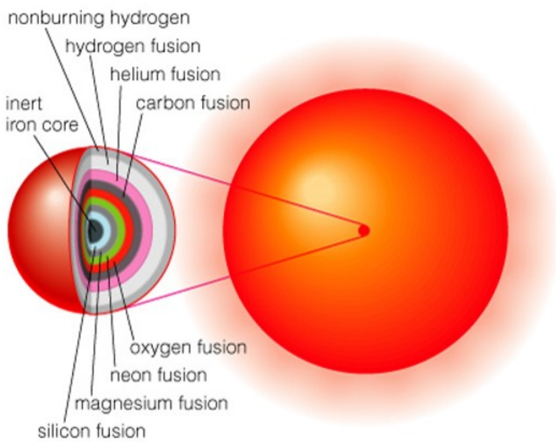
Expansion speeds of planetary nebulae are typically 10-30 $\mathrm{km\,s^{-1}}$. Sizes are $\sim 0.3\,\mathrm{pc}$. Lifetimes are just 10^4 years before a planetary nebula fades back into the interstellar medium.

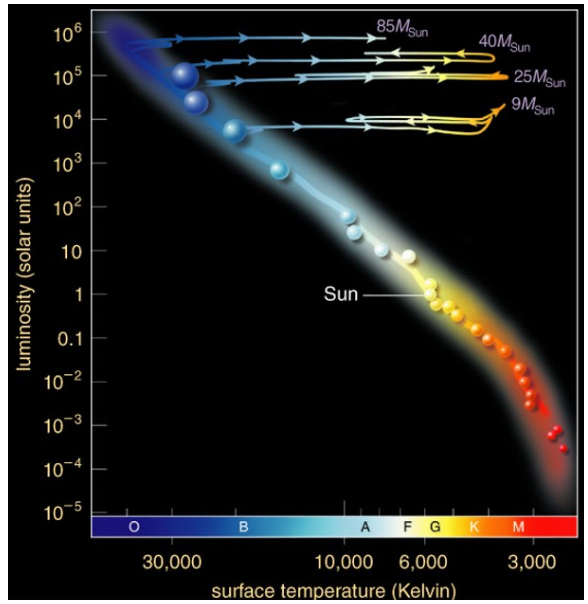
We now have a good understanding of what happens to stars of stellar mass $< 8\,M_\odot$. Stars with masses $> 8\,M_\odot$ initially follow a similar evolutionary sequence, but then they quickly diverge.

Massive Star Evolution

Lower mass stars end their lives with double-shell burning, building up a core composed of carbon and oxygen. If you remember the binding energy curve, elements up to iron can fuse to release energy. Why don’t low mass stars continue to fuse heavier elements? They do not have the requisite temperatures in their cores.

Massive stars, however, can continue fusing heavier and heavier elements until they reach iron. During this advanced fusion, the stars evolve off the main sequence, as do lower mass stars, although high-mass stars do so at about the same luminosity through all their later phases. They form successive layers of non-burning H, fusing H, fusing He, fusing C, fusing O, fusing Ne,





fusing Mg, fusing Si, and inert iron. You will notice that the atomic mass of the fusion elements in the inner shells differ from one another by atomic number 2. This is because the most efficient fusion reaction with these elements involves alpha particles.

This successive fusion produces less and less energy (look at binding energy curve). As a result, stars go through these evolutionary sequences really fast. Your book notes that if the MS lifetime is 10^7) years, He burning is 10^6 years, carbon burning is 300 years, oxygen burning is 200 days, and Si burning is only two days.

During this multiple-shell burning phase, the iron ash accumulates in the core. Ash accumulates in the other layers to but is burned off. Iron doesn't produce energy via fusion, and eventually the star goes supernova. More on supernovae later.

Types of Evolved Massive Stars

Before a star goes supernova, it goes through distinct evolutionary phases, many of which are named.

Luminous Blue Variables (LBVs)

LBVs are extremely massive, nearly 100 $M_{\odot}$. The most famous example is η Carinae. LBVs have tremendous mass loss of 10^{-3} $M_{\odot}$ per year (the Sun's is 10^{-14} $M_{\odot}$ per year). Some of this mass loss is explosive, as was the "Great Eruption" of η Car, which temporarily made it the second brightest star in the sky. η Car's mass loss has created the "homonculous," a massive cloud of gas

Periodic Table of the Elements

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1 H Hydrogen 1.008																	2 He Helium 4.003
3 Li Lithium 6.941	4 Be Beryllium 9.012											5 B Boron 10.811	6 C Carbon 12.011	7 N Nitrogen 14.007	8 O Oxygen 15.999	9 F Fluorine 18.998	10 Ne Neon 20.180
11 Na Sodium 22.990	12 Mg Magnesium 24.305											13 Al Aluminum 26.982	14 Si Silicon 28.086	15 P Phosphorus 30.974	16 S Sulfur 32.066	17 Cl Chlorine 35.453	18 Ar Argon 39.948
19 K Potassium 39.098	20 Ca Calcium 40.078	21 Sc Scandium 44.956	22 Ti Titanium 47.867	23 V Vanadium 50.942	24 Cr Chromium 51.996	25 Mn Manganese 54.938	26 Fe Iron 55.845	27 Co Cobalt 58.933	28 Ni Nickel 58.693	29 Cu Copper 63.546	30 Zn Zinc 65.38	31 Ga Gallium 69.732	32 Ge Germanium 72.631	33 As Arsenic 74.922	34 Se Selenium 78.971	35 Br Bromine 79.904	36 Kr Krypton 84.798
37 Rb Rubidium 84.468	38 Sr Strontium 87.62	39 Y Yttrium 88.906	40 Zr Zirconium 91.224	41 Nb Niobium 92.906	42 Mo Molybdenum 95.95	43 Tc Technetium 98.907	44 Ru Ruthenium 101.07	45 Rh Rhodium 102.906	46 Pd Palladium 106.42	47 Ag Silver 107.868	48 Cd Cadmium 112.414	49 In Indium 114.818	50 Sn Tin 118.711	51 Sb Antimony 121.760	52 Te Tellurium 127.6	53 I Iodine 126.904	54 Xe Xenon 131.294
55 Cs Cesium 132.905	56 Ba Barium 137.328	57-71 Lanthanides	72 Hf Hafnium 178.49	73 Ta Tantalum 180.948	74 W Tungsten 183.84	75 Re Rhenium 186.207	76 Os Osmium 190.23	77 Ir Iridium 192.217	78 Pt Platinum 195.085	79 Au Gold 196.967	80 Hg Mercury 200.592	81 Tl Thallium 204.383	82 Pb Lead 207.2	83 Bi Bismuth 208.980	84 Po Polonium [208.982]	85 At Astatine 209.987	86 Rn Radon 222.018
87 Fr Francium 223.020	88 Ra Radium 226.025	89-103 Actinides	104 Rf Rutherfordium [261]	105 Db Dubnium [262]	106 Sg Seaborgium [266]	107 Bh Bohrium [264]	108 Hs Hassium [269]	109 Mt Meitnerium [278]	110 Ds Darmstadtium [281]	111 Rg Roentgenium [280]	112 Cn Copernicium [285]	113 Nh Nihonium [286]	114 Fl Flerovium [289]	115 Mc Moscovium [289]	116 Lv Livermorium [293]	117 Ts Tennessine [294]	118 Og Oganesson [294]

57 La Lanthanum 138.905	58 Ce Cerium 140.116	59 Pr Praseodymium 140.908	60 Nd Neodymium 144.243	61 Pm Promethium 144.913	62 Sm Samarium 150.36	63 Eu Europium 151.964	64 Gd Gadolinium 157.25	65 Tb Terbium 158.925	66 Dy Dysprosium 162.500	67 Ho Holmium 164.930	68 Er Erbium 167.259	69 Tm Thulium 168.934	70 Yb Ytterbium 173.055	71 Lu Lutetium 174.967
89 Ac Actinium 227.028	90 Th Thorium 232.038	91 Pa Protactinium 231.036	92 U Uranium 238.029	93 Np Neptunium 237.048	94 Pu Plutonium 244.064	95 Am Americium 243.061	96 Cm Curium 247.070	97 Bk Berkelium 247.070	98 Cf Californium 251.080	99 Es Einsteinium [254]	100 Fm Fermium 257.095	101 Md Mendelevium 258.1	102 No Nobelium 259.101	103 Lr Lawrencium [262]

Alkali Metal | Alkaline Earth | Transition Metal | Basic Metal | Semimetal | Nonmetal | Halogen | Noble Gas | Lanthanide | Actinide

$$M > 85\,\mathrm{M}_\odot : \mathrm{O} \rightarrow \mathrm{Of} \rightarrow \mathrm{LBV} \rightarrow \mathrm{WN} \rightarrow \mathrm{WC} \rightarrow \mathrm{SN}$$

$$40\,\mathrm{M}_\odot < M < 85\,\mathrm{M}_\odot : \mathrm{O} \rightarrow \mathrm{Of} \rightarrow \mathrm{WN} \rightarrow \mathrm{WC} \rightarrow \mathrm{SN}$$

$$25\,\mathrm{M}_\odot < M < 40\,\mathrm{M}_\odot : \mathrm{O} \rightarrow \mathrm{RSG} \rightarrow \mathrm{WN} \rightarrow \mathrm{WC} \rightarrow \mathrm{SN}$$

$$20\,\mathrm{M}_\odot < M < 25\,\mathrm{M}_\odot : \mathrm{O} \rightarrow \mathrm{RSG} \rightarrow \mathrm{WN} \rightarrow \mathrm{SN}$$

$$10\,\mathrm{M}_\odot < M < 20\,\mathrm{M}_\odot : \mathrm{O} \rightarrow \mathrm{RSG} \rightarrow \mathrm{BSG} \rightarrow \mathrm{SN}$$

and dust that is expanding outward at 650 $\mathrm{km\,s^{-1}}$. https://www.youtube.com/watch?v=u5n
gSY
https://www.youtube.com/watch?v=07hqULmszC8

Wolf-Rayet (WR) Stars

WR stars are less massive that LBVs, maybe only 20 $M_\odot$. They are rapidly rotating and are losing mass at a rate of $> 10^{-5}$ $M_\odot$ per year.

Unlike most stars, WR stars have strong (and broad) emission line spectra. Where could the broadness come from? The emission line spectra have lines of He, C, N, and O. Where could these come from? The H atmosphere of WR stars is completely absent, having been stripped off. It is the processed layers that are revealed, and they are enriched. We can separate WR stars into the WN class, which has N and He emission lines, the WC class, which has C and He, and the WO class, which has O (and is quite rare).

Other Evolved High-Mass Star Types

$10-40$ $M_\odot$ stars will first evolve into Red SuperGiants (RSGs) immediately off the main sequence. These are luminous, but red.

$10-20$ $M_\odot$ stars will evolve through a Blue SuperGiant (BSG) phase that is characterized by high luminosities and (relatively) blue colors.

> 40 $M_\odot$ stars will evolve into Of stars, which are like O stars but have emission lines.

Supernovae (SN)

All good things must come to an end, and so too must a massive star's life end. In keeping with their "live fast, die young" mantra, they go out with a bang, as a SN.

There are type main types of supernova, designated (confusingly) as Type 1a and “Other” (other Type 1s and Type IIs). We will here be dealing with the Other category. These arise from core-collapse.

Eventually, fusion of Fe ceases, removing a source of pressure in the cores. Then the gravitational pressure becomes too great to be opposed by electron degeneracy pressure in the core, and the core will collapse in on itself. The layers just above the core are then drawn into a high temperature region and fusion takes place in these layers violently, leading to a SN explosion. These leave behind a supernova remnant, SNR. The SNR is the entirety of the star, minus the iron core which can turn into a neutron star or black hole. SNRs therefore enrich the interstellar medium. We are all made of stars!