Cosmology (Chapters 29 and 30)

Cosmology is the study of the nature of the Universe, including its chronology (its beginning, evolution, and eventual end). We have been introduced to some early versions of cosmology. For example, until the 1920s, astronomers didn't know if the Milky Way was the entire Universe.

Cosmology is the large-scale study of the evolution of the universe. A few hints about cosmology:
1) Olber's paradox says that if the universe was infinite in space and time, why do we not see a star along every line of sight? This after all is the situation for dense forests. One may be tempted to assume that intervening dust gets in the way, but in that case the dust would heat up as hot as a star and also glow since every line of sight would also contribute photons to its heating. The only solution is that the Universe is not old enough for the light to have reached us along every sight line. The Universe must have a finite age.

2) Edwin Hubble found that the more distant a galaxy is, the larger its recessional velocity. This is known as Hubble's Law, $v = H_0 d$, with H_0 being the Hubble constant, measured in km/s/Mpc. Hubble's Law implies that the Big Bang happened, because we can rewind time back to when all galaxies existed in the same place.

Now we will begin to develop cosmology more scientifically. We start with the **Cosmological Principle**: the Universe is homogeneous (same everywhere; the part of the universe we can see is a fair sample) and isotropic (same in all directions; same physical laws apply throughout). Viewed on a sufficiently large scale, the properties of the universe are therefore the same for all observers. In other words, the part of the universe that we can see is a fair sample, and that the same physical laws apply throughout. In essence, this says that the universe is knowable and is playing fair with scientists. This idea was first stated clearly by Isaac Newton in *Principia*. A corollary to this is that we do not live in a special place in the Universe.

Universe Expansion

The Universe is expanding, and quickly. In the early 1900s, V.M. Slipher was measuring the redshift of galaxies. These galaxies were not known to be external to the Milky Way, so it was especially surprising when he found that all of them, with the exception of Andromeda, were redshifted. How does this work?

$$\frac{(\lambda_o - \lambda_e)}{\lambda_e} = \frac{\Delta\lambda}{\lambda} = v/c = z \tag{1}$$

or

$$\frac{\nu_e - \nu_o}{\nu_o} = \frac{\Delta\nu}{\nu} = \frac{v}{c} = z \tag{2}$$

where the "o" subscripts refer to the observed and "e" to emitted and z is the redshift ($z = v/c$). Here we see wavelength or frequency shifts related to the recessional velocity. If the wavelength is decreasing or the frequency is increasing, the velocity is positive. Note the difference in the order of subtraction. The numerators are a matter of convention for optical astronomy. These are the non-relativistic formulae, for when $v \ll c$.

In the relativistic limit,

$$\frac{\lambda_o}{\lambda_e} = \left(\frac{1+\beta}{1-\beta}\right)^{0.5} - 1 \tag{3}$$

or

$$z = \left(\frac{1+\beta}{1-\beta}\right)^{0.5} - 1 \tag{4}$$

or

$$\frac{v}{c} = \left(\frac{(z+1)^2 - 1}{(z+1)^2 + 1}\right) \tag{5}$$

where $\beta = v/c$.

Note that to actually measure redshift we need multiple identifiable spectral lines.

In 1925, Hubble discovered Cepheids in Andromeda, and established that it was external to the Milky Way. Identifying Cepheids in other galaxies, and combining with Slipher's velocities, he found that the redshift was correlated with the distance:

$$v = H_0 d\,, \tag{6}$$

where H_0 is the Hubble constant. This is Hubble's Law. Since v is in $\mathrm{km\,s^{-1}}$ and d in Mpc, H_0 is in $(\mathrm{km\,s^{-1}})/\,\mathrm{kpc}$. We can therefore reduce it to units of 1/s.

Hubble understood that he had found evidence for the expansion of the universe. This was the birth of modern cosmology.

What does Hubble's Law mean? As long as we are not in a special place in the Universe (more on that later), the expansion must happen everywhere at once. Imagine the Universe as the United States. If it doubled in size in one second, from our vantage point locations 1000 km away would be 2000 km away, and moving away at 1000 $\mathrm{km\,s^{-1}}$. Locations 2000 km away would now be 4000 km away and moving away at 2000 $\mathrm{km\,s^{-1}}$. We therefore see the expected relationship between v and d. This expansion does not affect gravitationally bound systems like galaxies themselves, the Solar System, galaxy clusters, etc.

One of the real powers of using Hubble's law is that we have another way of determining distances. In the non-relativistic case,

$$d = \frac{cz}{H_0} \tag{7}$$

This should only be used for $z < 0.13$. Or for higher redshifts,

$$d \simeq v/H_0 = \frac{c}{H_0}\frac{(z+1)^2-1}{(z+1)^2+1} \tag{8}$$

The value of H_0

H_0 is simply the slope of the v vs d line. Why could that be? As we saw above, it's tough to measure d. So we can parameterize our ignorance:

$$H_0 = 100h\,(\mathrm{km\,s^{-1}})/\mathrm{Mpc} \tag{9}$$

Hubble Time

We can move time backwards to determine an approximate age of the Universe. If the Universe is expanding at H_0, we can shrink it from its current size back to a single point, by H_0. The time for this to happen is the Hubble time, t_H. We can find the Hubble time by taking $1/H_0$. If the Universe expanding at the same rate throughout time, the Hubble time is the age of the Universe.

Value of H_0

The Hubble constant is near 70 $\mathrm{km\,s^{-1}}$/Mpc, although different experiments find different values. The Planck satellite found H_0 is 67.3 ± 1.2 $\mathrm{km\,s^{-1}}$/Mpc.

There are many ways that astronomers have devised to determine H_0. There are two main methods:

- Satellites can measure the fluctuations in the "Cosmic Microwave Background" (CMB) radiation. We'll talk about the CMB a lot later.

- We can measure the redshift of distant galaxies and then determine the distances to the same galaxies (by some method other than Hubble's law). The slope of the d vs. v relationship is H_0.

We will discuss the first method later. The second method is best measured with Type 1a supernova remnants. Both the Super Cosmology Project and the High-Z Supernova Search Team, another team who was doing the same research, expected to find that the universe is either expanding then contracting as one way to explain the expanding universe idea or the universe must be expanding at a slow rate that will slow over time. However, in January 1998, the Supernova Cosmology project presented evidence that the expansion of the universe is not slowing at all and is in reality accelerating. In 2011, Riess and Schmidt of the Super Cosmology Project, along with Saul Perlmutter of the Supernova Cosmology Project, were awarded the Nobel Prize in Physics for this work.

The argument for this finding is a bit difficult, and we'll return to it later but let's walk through it here. For each supernova, we have an expectation that the absolute magnitude is -19.3. We measure the apparent magnitude and use the distance modulus to get the distance. We simultaneously measure the redshift and can therefore make a plot of distance modulus $(m - M)$ vs reshift. What does constant expansion look like on this graph? Decelerating? Accelerating?

Day 2: A Model Universe

Topics: k, R, critical density ρ_c, density parameter Ω, geometries (flat, open, or closed)

Let's develop a simple model for an expanding universe. If the cosmological principle is valid, we can replace matter in the universe with simple, pressureless "dust." This dust is not the same as ISM dust, but is simply a way to illustrate the homogeneous matter distribution. There are no photons or neutrinos, just pressureless matter.

Imagine a thin expanding shell of this dust of radius $r(t)$ and mass m representing expansion of the universe. The expansion speed is $v(t) = dr(t)/dt$. The total energy of the shell is constant. Therefore,

$$K(t) + U(t) = E \tag{10}$$

$$\frac{1}{2}mv^2(t) - G\frac{M_r m}{r(t)} = -\frac{1}{2}mkc^2\varpi^2 \tag{11}$$

Here M_r is the mass interior to the shell. The kinetic energy is positive (expansion) and is resisted by negative potential energy. The total energy of the shell has a very strange form, and the additional variables k (dimensions length^{-2}) and ϖ (dimensions length). We can think of k as describing the geometry of the Universe and ϖ as the current radius of the shell, so $r(t_0) = \varpi$. The negative sign in the potential energy is because of our convention for k.

The interior mass M_r must remain constant, so

$$M_r = \frac{4}{3}\pi r^3 \rho(t) \tag{12}$$

We can cancel m and substitute for M_r to get

$$v^2 - \frac{8}{3}\pi G\rho r^2 = -kc^2\varpi^2 \tag{13}$$

This is a profound result. If k is:

0: the expansion will continually slow down, asymptotically approaching an expansion velocity of zero. It continues the initial expansion because of inertia. This is a **flat** universe.

< 0: the overall energy is positive and the universe is unbounded - the expansion continues forever. This is an **open** Universe.

> 0: the overall energy is negative and the universe is bounded - expansion will stop and reverse itself. This is a **closed** Universe.

We can write that

$$r(t) = R(t)\varpi \tag{14}$$

where $R(t)$ is the **scale factor** that describes the expansion[1]. This makes sense because ϖ is the present radius of the shell, and we define $R = 1$ at present. This scale factor is telling you about the size of the Universe. If $R > 1$, the Universe is larger than the present day, and if $R < 1$, the Universe is smaller. We can relate the scale factor to redshift:

$$R = \frac{1}{1+z} \tag{15}$$

For a pressureless universe, as the Universe expands, the total mass remains constant. Therefore, the quantity

$$R^3(t)\rho(t) = R_0^3(t)\rho_0(t) = \rho(t) \tag{16}$$

and

$$\rho(z) = \rho_0(1+z)^3 \tag{17}$$

This is valid only for our pressureless Universe, but the result is still a good one.

Evolution of Pressureless Universe We now want to see how our pressureless universe evolves with time. We can begin with Hubble's Law in a slightly revised form

$$v(t) = H(t)r(t) = H(t)R(t)\varpi \tag{18}$$

but

$$v(t) = \frac{dR(t)}{dt}\varpi \tag{19}$$

so

$$H(t) = \frac{1}{R(t)}\frac{dR(t)}{dt} \tag{20}$$

We can plug this in to our earlier expression for the energy balance, cancel the ϖ^2 term and get

$$\left(H^2 - \frac{8}{3}\pi G\rho\right)R^2 = -kc^2 \tag{21}$$

A very important quantity comes directly out of this equation, that of the **critical density** when $k = 0$. This is the exact amount of matter needed to balance the expansion.

$$\rho_c(t) = \frac{3H^2(t)}{8\pi G} \tag{22}$$

We see that the critical density evolves with time. The present value of the critical density

$$\rho_{c,0} = \frac{3H_0^2}{8\pi G} \tag{23}$$

We can define a further important parameter, that of the **density parameter**

$$\Omega = \frac{\rho(t)}{\rho_c(t)} = \frac{8\pi G\rho}{3H^2} \tag{24}$$

which has a present value of

$$\Omega_0 = \frac{\rho_0(t)}{\rho_{c,0}(t)} = \frac{8\pi G\rho_0}{3H_0^2} \tag{25}$$

[1] Some texts use a for the scale factor; we'll follow the book's notation here

Therefore the density parameter tells us how close we are to the critical density. Finally, we can rewrite our energy equation as

$$\left(H^2 - \frac{8}{3}\pi G\rho\right) R^2 = -kc^2 \tag{26}$$

$$H^2(1-\Omega)R^2 = -kc^2 \tag{27}$$

or at present

$$H_0^2(1-\Omega_0) = -kc^2 \tag{28}$$

Let's now revisit our flat, open, and closed universe models. An interesting and important aspect of the above equations is that Ω_0 determines the geometry k, and k is not a function of time. We can therefore use Ω_0 to define the geometry.
If $\Omega = 1$, the Universe is flat ($k = 0, \rho = \rho_c$).
If $\Omega < 1$, the Universe is open ($k < 0, \rho < \rho_c$)
$\Omega > 1$, the Universe is closed ($k > 0, \rho > \rho_c$)

Day 3

These three geometries evolve in very different ways. If we take

$$H = \frac{1}{R}\frac{dR}{dt}\,, \tag{29}$$

and $k = 0$ then

$$\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G\rho_{c,0}}{3R} \tag{30}$$

This equation can be solved to get

$$R_{\text{flat}} = (6\pi G\rho_{c,0})^{1/3} t^{2/3} \tag{31}$$

$$= \left(\frac{3}{2}\right)^{2/3}\left(\frac{t}{t_H}\right)^{2/3} \tag{32}$$

The derivation is significantly more complicated if $\Omega_0 \neq 1$, but I can show you the solution.

$$R_{\text{closed}} = \frac{4\pi G\rho_0}{3kc^2}[1-\cos(x)] \tag{33}$$

$$= \frac{1}{2}\frac{\Omega_0}{\Omega_0 - 1}[1-\cos(x)] \tag{34}$$

$$t_{\text{closed}} = \frac{4\pi G\rho_0}{3k^{3/2}c^3}[x-\sin(x)] \tag{35}$$

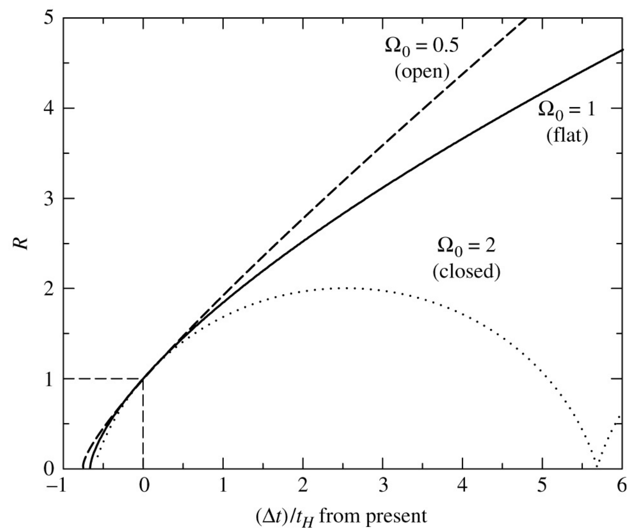
$$= \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}[x-\sin(x)]\,. \tag{36}$$

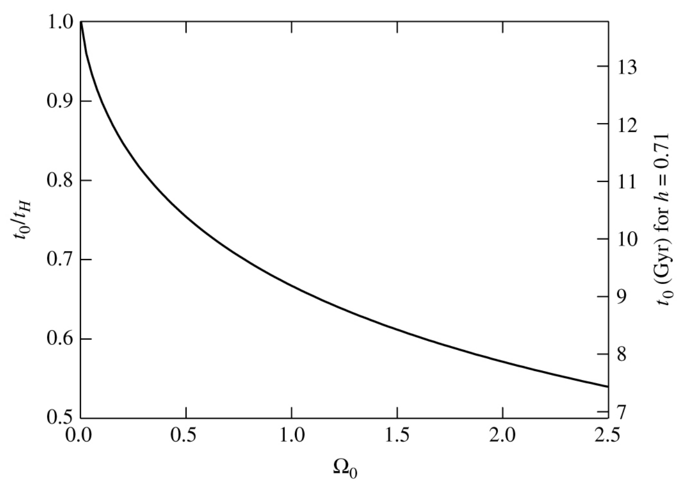
$$R_{\text{open}} = \frac{4\pi G\rho_0}{3|k|c^2}[\cosh(x)-1] \tag{37}$$

$$= \frac{1}{2}\frac{\Omega_0}{1-\Omega_0}[\cosh(x)-1] \tag{38}$$

$$t_{\text{open}} = \frac{4\pi G\rho_0}{3|k|^{3/2}c^3}[\sinh(x)-x] \tag{39}$$

$$= \frac{1}{2H_0}\frac{\Omega_0}{(\Omega_0-1)^{3/2}}[\sinh(x)-x]\,, \tag{40}$$





where x parameterizes the solution.

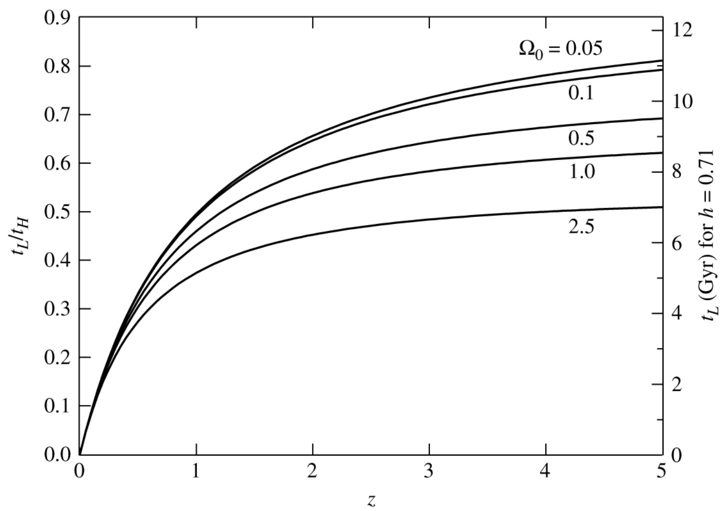
We see that all model Universes have the same scale factor of 1 for the present day, and that there is little variation in the early Universe. At later times, the Universes will diverge greatly. The closed Universe "bounce" is only a mathematical effect. Note that each line crosses the x-axis at different x-values, leading to different ages of the Universe, with open being the longest, then flat, then closed.

In addition to different evolutions of the scale factor, the Universe geometries determine the age and time-evolution of the Universe. We can take the above equation and substitute in $R = 1/(1+z)$ to get

$$\frac{t_{\text{flat}}}{t_H} = \frac{2}{3}\frac{1}{(1+z)^{3/2}} \tag{41}$$

This is interesting! At the present day, the age of a flat, pressureless dust Universe is $2/3$ of the Hubble time. We know already that the Hubble time is approximately the age of the Universe, so obviously either the Universe is not flat, or pressure is important. As we will see, photon pressure in particular is important at some times of the Universe evolution.

Lookback Time The Lookback Time is how far in the past we are seeing when we observe something at



redshift z. We can say that $t_L = t_0 - t(z)$. Therefore, for a flat Universe,

$$\frac{t_L}{t_H} = \frac{2}{3}\left[1 - \frac{1}{(1+z)^{3/2}}\right] \tag{42}$$

We see that for a flat Universe, if $z = 1$, the lookback time is about half a Hubble time. As before, the other solutions are more complicated, but they are easy to understand from the graphs. As the Universe has more material (Ω_0 larger), the lookback time becomes smaller for a given redshift. This is another way of saying that the expansion is slowing.

Including Pressure

We can include pressure in our simple model to arrive at a much more realistic treatment. This is in principle quite simple. If instead of the mass density, ρ is the "equivalent mass density." There is a relationship between "state variables" used to define the state of a thermodynamic system; for our purposes pressure and density. This is called an "equation of state." The simplest equation of state is the ideal gas law:

$$PV = NkT\frac{P}{\rho} \propto T \tag{43}$$

We can express an equation of state more generally if we remember that the pressure is proportional to the energy density. We can then write

$$P = wu = w\rho c^2 \tag{44}$$

where u is the energy density and we have substituted in to get a term that looks like $E = mc^2$. The parameter w is the important one here. We know that $w = 0$ for pressureless dust and $w = 1/3$ for backbody radiation (since $P_{\rm rad} = u_{\rm rad}/3$).

We have one final parameter: the **deceleration parameter** q:

$$q(t) = -\frac{R(t)[d^2R(t)/dt^2]}{[dR(t)/dt]^2} \tag{45}$$

So $q > 0$ for a decelerating Universe and $q > 0$ for an accelerating universe. For a pressureless dust Universe, you can show that

$$q(t) = \frac{1}{2}\Omega(t) \tag{46}$$

Whew! OK, so let's review all of our cosmological parameters:

- $H(t)$, the Hubble constant, very closely related to $R(t)$ and z. At present, $H(t) = H_0$
- z, the redshift, related to $H(t)$, $R(t)$, and t_L. At present, $z = 0$ and z increases toward the Big Bang.
- $R(t)$, the scale factor, or "size" of the Universe, related to above quantities. At present, $R = 1$ and R is smaller closer to the Big Bank when the Universe was smaller.
- t_L, the lookback time. Related to z and Ω_0.
- ρ_c, the critical density necessary to halt the expansion of the Universe at $t = \infty$.
- Ω_0, the density parameter, tells us how close to the critical density we are. $= 1$ for flat, < 1 for open, and > 1 for closed.
- k, which we will call the "curvature parameter", =0 for flat, < 0 for open, > 0 for closed.
- w, the equation of state term used to relate pressure and density. $w = 0$ for pressureless dust and $w = 1/3$ for blackbody radiation.
- q, the deceleration parameter.

So these are the parameters we need to determine if we are to understand our Universe. Since they are all interrelated, we actually only need to determine H_0, Ω_0, and w. What do we actually measure? These values are from Planck:

- $H_0 = 67.8 \pm 0.9$ km/s/Mpc
- $\Omega_0 \simeq 1$ (flat Universe)
- $w = -1$ This means a constant energy density, and negative pressure... (more on this later!)

The value for w hints that this is only part of the story though, because there is physics not yet discussed....

Revisions to this derivation I will write the most important equation again in two ways:

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho\right] R^2 = -kc^2 \tag{47}$$

This is known as the Friedmann equation. From last time, we found that for ρ we can use an "equivalent mass density" if we have an equation of state. This can be simplified to

$$H^2[1-\Omega]R^2 = -kc^2 \tag{48}$$

It is shocking (to me) that solving Einstein's equations of general relativity led Aleksandr Friedmann to derive this equation in 1922 when solving for an isotropic, homogeneous universe (also derived by Lemaitre independently). (BTW, Lemaitre was the first to recognize that the equations imply a "Big Bang.") So our simple derivation of the evolution of a thin dust sphere is also valid for GR. We do have to change our understanding of k though. More on that later.

Problems with this:
1) We need to explicitly include radiation pressure. We can do this easily by defining $\rho_{\rm rel}$ for relativistic particles (photons). We then have ρ_m for matter, and in the Friedmann equation $\rho = \rho_{\rm rel} + \rho_m$. Therefore

$$\Omega_{\rm rel} = \rho_{\rm rel}/\rho_{\rm c} = \frac{8\pi G\rho_{\rm rel}}{3H^2} \tag{49}$$

Similarly, for matter,

$$\Omega_{\rm m} = \rho_{\rm m}/\rho_{\rm c} = \frac{8\pi G\rho_{\rm m}}{3H^2} \tag{50}$$

and therefore,

$$H^2[1 - (\Omega_m + \Omega_{\rm rel})]R^2 = -kc^2 \tag{51}$$

2) This equation does not result in a "steady state" solution! In other words, the Universe is continually expanding. We can see this from our plot of the time evolution of R.

In 1917, Einstein realized that the Friedmann equation cannot be solved to yield a static Universe. At the time, no one thought that the Universe was expanding, as Hubble's work was not yet done. So Einstein input a "Cosmological constant" Λ so that the Friendmann equation yields a steady-state solution (he is often quoted as saying this was is "greatest blunder", although there is some doubt as to whether this is correctly attributed.):

$$\left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\rho - \frac{1}{3}\Lambda c^2\right] R^2 = -kc^2 \tag{52}$$

This "fudge factor" allows the equation to balance so that the expansion is zero. Notice that if Λ is positive, the term has negative energy, and the energy is a function of R^2. Since the force

$$F = -\frac{dE}{dr}\,, \tag{53}$$

$F_\Lambda \propto +R$, which is positive. The larger the scale factor, the larger the force. The Cosmological constant provides a positive expansion! Another way to think about it is a negative pressure, or a negative energy density. This is called "dark energy," a repulsive force that opposes gravity. Notice that since Λ is a constant, it is uniform everywhere (which is generally accepted). As best we can tell, it is the cost we pay for having spacetime - it is baked into the fabric of the Universe. Huh.

How can we understand dark energy? It may be related to the Casimir effect. In the Casimir effect, there is an attractive force between two parallel uncharged conuctive plates in a vacuum. While classically there is no field between the plates, quantum electrodynamics predicts that there is one. This explains the observed effect that the plates move together in the absence of any other forces. This negative energy density is also predicted by particle physics. Unfortunately, quantum electrodynamics and particle physics disagree on the magnitude of the effect by over 100 orders of magnitude. Even for astronomers, this is too big....

We can define additional terms as before:

$$\Omega_\Lambda = \rho_\Lambda/\rho_{\rm c} = \frac{8\pi G\rho_\Lambda}{3H^2} \tag{54}$$

and therefore,

$$H^2[1 - (\Omega_m + \Omega_{\rm rel} + \Omega_\Lambda)]R^2 = -kc^2 \tag{55}$$

The total density paramter

$$\Omega = \Omega_m + \Omega_{\rm rel} + \Omega_\Lambda \tag{56}$$

so we once again have

$$H^2[1 - \Omega]R^2 = -kc^2 \tag{57}$$

This is exactly the same as our previous expression. Only the definition of Ω has changed.

But there is one strange thing about dark energy. We see from the revised Friedmann equation that in order for ρ_Λ to have the same form as the other densities, we must write

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G} \tag{58}$$

and if $\Lambda = \text{const}$, then $\rho_\Lambda = \rho_{\Lambda,0}$. This is profound! The density of dark matter never decreases, so as the Universe expands, more and more dark energy appears to fill the increasing volume. This is what I meant when I said that it was baked into the fabric of the Universe. Huh.

3) What is k? The GR solution in the Friedmann equation shows that k is related to the geometry of the Universe. In normal Euclidean geometry a straight lines parallel to another line will continue parallel infinitely and never intersect. This is known as Euclid's fifth postulate. In the nineteenth century, mathematicians found that there are other geometries that are self-consistent. For these geometries there exists
(a) no line parallel to the first (all lines will eventually intersect) and
(b) at least two lines parallel to the first. These are known as "elliptic" and "hyperbolic" geometries.

So the only loose end left is to understand that the elliptic and hyperbolic geometries are related to the curvature constant in that:
$k = 0$: flat, Euclidean geometry with no curvature
$k > 0$: closed, elliptic geometry with positive curvature
$k < 0$: open, hyperbolic geometry with negative curvature

Notice that this is still the Newtonian classical solution. The full GR treatment gives (amazingly) the same result. The only difference is the interpretation, with GR providing the information that k is actually related to the geometry.

The History of the Universe

The Universe began with the Big Bang. (This term was coined by Fred Hoyle in 1950.) It started out as a single point of infinite temperature and density, analogous to a black hole singularity, and has been expanding until the present day. I want to discuss the time-evolution of the Universe by defining some of the relevant epochs, and then by delving deeper into each epoch.

The main point in the evolution of the Universe from the Big Bang is that the temperature was hot and is decreasing, and the density was high and is decreasing. This drives most of the rest of the evolution.

The first epoch is called the Planck epoch. The only way one can combine the physical constants into something with the dimensions of time is

$$t_P = \left(\frac{\hbar G}{c^5}\right)^{0.5} \tag{59}$$

This is the "Planck Time", which is 10^{-43} s, and we cannot know about anything before this time.

After the Planck epoch comes the Grand unification epoch. In the Planck epoch, the four forces were combined into one force. When gravity separates, the Grand Unification epoch begins.
$10^{-43} - 10^{-36}$ s.

In the Electroweak Epoch, the strong force separated from the electroweak force. It is in this epoch that we have "inflation," which we will discuss later.
$10^{-36} - 10^{-32}$ s

Eventually the Universe cools enough to start forming particles. What was there before? Just photons! At sufficient energies the Universe is too hot to have stable particles. Instead, particles pop in and out of existence. A high energy photon has a non-zero probability of becoming a particle-antiparticle pair, which then annihilate to form the photon again. These were the conditions in the early Universe.

Hadron (incl. protons and neutrons) epoch
Between 10^{-6} second and 1 second after the Big Bang

Lepton (inc. electrons) epoch
Between 1 second and 10 seconds after the Big Bang

Photon epoch
Between 10 seconds and 380,000 years after the Big Bang
The Universe at this point is still just a soup of particles and photons. In fact, it is so dense, that it is optically thick. Within the photon epoch are:

Nucleosynthesis
Between 3 minutes and 20 minutes after the Big Bang
This is the most important epoch. All the elements are created in nuclear reactions. By "elements" I just mean nuclei at this stage.

Recombination
380,000 years after the Big Bang
At some point the Universe had expanded to the point where it was optically thin. At this point it is still a soup of particles. The high photon energy and interaction rate means that everything is ionized. When the Universe becomes optically thin, the photons are free to travel and the fewer interactions between photons and matter allow matter to form atoms. This is confusingly called "recombination" although it was never combined previously.

The Dark Ages
Between 380,000 years and maybe 150 MYr after the Big Bang. At this point there was neutral hydrogen, but nothing else. Galaxies had not yet formed. There were no stars. There are numerous experiments ongoing to detect the HI signal from this era.

Reionization
Maybe 150 million to 1 billion years after the Big Bang
Eventually the first stars do form. This "reionizes" the Universe because the first stars were quite massive.

How do we know there was a Big Bang?

CMB anisotropy:
Because of the Sun's motion with respect to the Hubble flow:

$$T_{\text{moving}} = \frac{T_{\text{rest}}(1 - v^2/c^2)^{0.5}}{1 - (v/c)\cos\theta} \tag{60}$$

where θ is the angle between the direction of observation of the CMB and the direction of motion. This reduces to

$$T_{\text{moving}} \simeq T_{\text{rest}}\left(1 + \frac{v}{c}\cos\theta\right) \tag{61}$$

for velocities much less than c.

There are peculiar motions of the Sun around the Galactic center, and also for the Local Group that sum to about 627 km/s toward the constellation Hydra.

Inflation and the solution to some of the problems with the standard model
There are a number of problems with the standard model:
1) Why is the CMB so smooth? The Universe expanded fast early in its history, and photons more than 2 degrees on the sky today were not in causal contact previously. So how could the temperature of the CMB be so smooth? This is the horizon problem.
2) Why is the Universe so flat? Observationally $\Omega_0 = 1$, but why? If Ω_0 were slightly different, we wouldn't have stars and galaxies. That would be bad. This is the "flatness" problem."

3) It's not obvious, but the standard model predicts a number of magnetic monopoles. These have not beed found.

Inflation Inflation solves the horizon and flatness problem. The inflation theory is basically accepted today, and says that the initial Universe did not have $\Omega = 1$. Inflation rapidly expanded the Universe. This solves the flatness problem because initially everything was close together, and it was causally connected. Then it expanded. It solves the flatness problem because it acts in such a way to stretch spacetime such that today $\Omega_0 = 1$, but it wasn't originally. The reason for inflation is still being investigated.

Why is there structure in the Universe?
Just as there is no obvious Talk about power spectrum?

Three Eras in the History of the Universe
We know that the Universe has cooled and decreased in density. Now let's figure out how.

Some of the analysis we skipped results in the relationship between the scale factor, the equation of state parameter, and the density of the Universe:

$$R^{3(1+w)}\rho = constant = \rho_0 \tag{62}$$

This ρ can be mass or energy density. Let's start early in the Universe when there was only relativistic particles. This is the **radiation era**. Therefore, for radiation:

$$R^{3(1+w_{\rm rad})}u_{\rm rad} = R^4 u_{\rm rad} = u_{\rm rad,0} \tag{63}$$

since for blackbody radiation $w = 1/3$ (and the CMB is a perfect blackbody). We could similarly write

$$R^4 \rho_{\rm rel} = u_{\rm rel,0} \tag{64}$$

For blackbody radiation, the energy density

$$u = aT^4 \tag{65}$$

so therefore

$$R^4 aT^4 = aT_0^4 \tag{66}$$

and

$$RT = T_0\,. \tag{67}$$

Therefore, we can determine the temperature of the CMB at any point in the history of the Universe.

When matter dominates, we'll call it the **matter era.** When matter is no longer being created, the total mass must be constant, so

$$R^3 \rho_m = \rho_{m,0} \tag{68}$$

When is the transition? When $\Omega_m = \Omega_{\rm rel}$, or $\rho_m = \rho_{\rm rel}$.

$$\frac{\rho_{\rm rel,0}}{R^4} = \frac{\rho_{\rm m,0}}{R^3} \tag{69}$$

so

$$R_{\rm rel,m} = \frac{\rho_{\rm m,0}}{\rho_{\rm rel,0}} = \frac{\Omega_{\rm m,0}}{\Omega_{\rm rel,0}} \tag{70}$$

We know that $\rho_{\rm rel} = g_* aT^4$, with a being the radiation constant and g_* the effective degrees of freedom for relativistic particles, and so by measuring T for the CMB we get a relationship between R and $\Omega_{m,0}$.

There may then be a transition to a dark energy dominated Universe. When would that take place? We know that the dark energy density is a constant, so

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G} = constant = \rho_{\Lambda,0} \tag{71}$$

or

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H^2} \tag{72}$$

Notice that there is no dependence on R here! Thus, when $\Omega_m = \Omega_\Lambda$, this leads to

$$\Omega_{m,0} R^{-3} = \Omega_{\Lambda,0} \tag{73}$$

or

$$R_{m,\Lambda} = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}} \right)^{1/3} \tag{74}$$

So now we have three potential eras. When does the transition between these eras take place, and what is actually measured? As with many things, it's not quite as easy to explain as we'd like. The favorite model of cosmology, and what we have been discussing, is the ΛCDM model. That stands for a Universe that has dark energy, and "cold" dark matter (i.e., not neutrinos). This model makes specific predictions for how the CMB should look, namely the appearance of something called the CMB "power spectrum," which tells us how much power is contained at various angular scales.

If the CMB temperature fluctuations are smooth, there will be little power at small spatial scales. If the fluctuations are all small with nothing large-scale, there will be more power at small spatial scales.

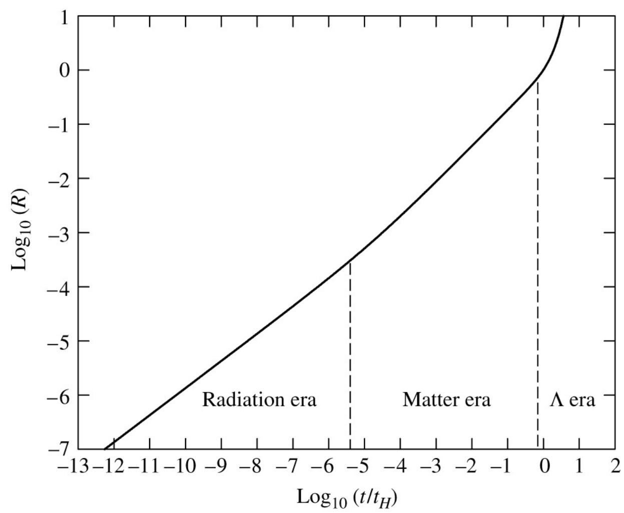
The power spectrum is important for two reasons:
1) Measurements of the CMB allow us to fit models to the derived power spectrum, and therefore derive various cosmological parameters, namely Ω_m.
2) The small lower-temperature structures in the CMB represent the "seeds" of structure in our present-day Universe. Those slightly lower temperature patches have slightly higher density and they grew into today's superclusters.

See how things change here: http://background.uchicago.edu/

So we measure the power spectrum, and derive Ω_m and Ω_Λ. With these values in hand, we can then determine when the Universe transitioned from radiation to matter to dark energy dominated (see previous equations). These transitions happened at

$$R_{rel,m} = 3.05 \times 10^{-4} \tag{75}$$
$$z_{rel,m} = 3230 \tag{76}$$
$$t_{rel,m} = 55,000 \text{ years} \tag{77}$$
$$R_{m,\Lambda} = 0.72 \tag{78}$$
$$z_{m,\Lambda} = 0.39 \tag{79}$$
$$t_{rel,m} = 7 \times 10^9 \text{ years} \tag{80}$$

So the transition from radiation to matter dominated happened fast, but the transition from matter to dark energy dominated took place only in the last half of the Universe's evolution.



The book gives a nice derivation of the full time-evolution of the scale factor (Equation 29.131). In short, during the radiation era, $R \propto t^{1/2}$, during matter-dominated $R \propto t^{2/3}$ and during dark energy-dominated R grows exponentially.

We can determine how the density parameters change.

$$\Omega_i(t) = \frac{8\pi G}{3H^2(t)}\rho(t) \tag{81}$$

and after a bit of algebra we find

$$\dot{\Omega_i}(t) = \Omega_i \left(\frac{\dot{\rho}_i(t)}{\rho_i(t)} - \frac{2\dot{H}(t)}{H(t)} \right) \tag{82}$$

We have an expression for H that depends on Ω and expressions for $\rho(R)$ for all species.

The Ultimate Fate of the Universe

We have already discussed what will happen to the Universe, but let's review some other possibilities and talk about further evidence for the ultimate fate.

Big Rip: $>$ 20 billion years from now
This scenario is possible only if the energy density of dark energy actually increases without limit over time. In this case, the expansion rate of the universe will increase without limit. Gravitationally bound systems, such as clusters of galaxies, galaxies, and ultimately the Solar System will be torn apart. Eventually the expansion will be so rapid as to overcome the electromagnetic forces holding molecules and atoms together. Finally even atomic nuclei will be torn apart and the universe as we know it will end in an unusual kind of gravitational singularity. At the time of this singularity, the expansion rate of the universe will reach infinity, so that any and all forces (no matter how strong) that hold composite objects together (no matter how closely) will be overcome by this expansion, literally tearing everything apart.

Big Crunch: $>$ 100 billion years from now
If the energy density of dark energy were negative or the universe were closed, then it would be possible that the expansion of the universe would reverse and the universe would contract towards a hot, dense state. Current observations suggest that this model of the universe is unlikely to be correct, and the expansion will continue or even accelerate.

Big Freeze: $> 10^5$ billion years from now
This scenario is generally considered to be the most likely, as it occurs if the universe continues expanding as it has been. Over a time scale on the order of 10^{14} years or less, existing stars burn out, stars cease to be created, and the universe goes dark. Over a much longer time scale in the eras following this, the galaxy evaporates as the stellar remnants comprising it escape into space, and black holes evaporate via Hawking radiation. In some grand unified theories, proton decay after at least 10^{34} years will convert the remaining interstellar gas and stellar remnants into leptons (such as positrons and electrons) and photons. Some positrons and electrons will then recombine into photons. In this case, the universe has reached a high-entropy state consisting of a bath of particles and low-energy radiation. It is not known however whether it eventually achieves thermodynamic equilibrium.

Heat Death: 10^{1000} years from now
The heat death is a possible final state of the universe, in which it has "run down" to a state of no thermodynamic free energy to sustain motion or life. In physical terms, it has reached maximum entropy (because of this, the term "entropy" has often been confused with heat death, to the point of entropy being labelled as the "force killing the universe"). The hypothesis of a universal heat death stems from the 1850s ideas of William Thomson (Lord Kelvin) who extrapolated the theory of heat views of mechanical energy loss in nature, as embodied in the first two laws of thermodynamics, to universal operation.

How can we determine the fate of the Universe?
We need some way to measure the change in H. The Friedmann equation shows that the value of H is related to the redshift. Rearranging terms in the Friedmann equation, we get

$$H = H_0(1+z)\left[\Omega_{m,0}(1+z) + \Omega_{rel,0}(1+z)^2 + \Omega_{\Lambda,0}(1+z)^{-2} + 1 - \Omega_0\right]^{1/2} \tag{83}$$

This is Eqn. 29.122. This gives the evolution of the Hubble parameter with redshift. BUT, notice that the overall geometry of the Universe matters a lot, and as z decreases, Ω_Λ becomes more important.

This is profound! The expansion of the Universe depends on the redshift in a measurable, predictable way.

So all we need to do is measure the distance to a set of objects, then measure how fast they are receding from us. If H has always been constant, there will be no difference in these two methods, even at large z.
0) We need a "standard candle." Cepheids are too dim. Need Type 1a supernovae
1) Find supernova Type 1a (hard! Rare!)
2) Measure their brightnesses
3) From the brightnesses (and known luminosities), calculate distances
4) Measure their redshifts, which provides another estimate of distance from Hubble's law
5) Compare these two distance estimates.

But there is one complication. The distance inferred from the brightness measurements is actually called the "luminosity distance" (see below).

$$d_L = \frac{L}{4\pi F}\,, \tag{84}$$

with L and F as luminosity and flux. Imagine a source emitting one photon per second. At cosmological distances we have a cosmological redshift that reduces the wavelength by a factor of $(1+z)$. There is also cosmological time dilation that reduces the time between photons received. This effect adds another factor of $(1+z)$.

The full derivation of d_L requires GR, unfortunately, and does not result in a nice analytical solution. Crap. We can still use the result.

We know that

$$m - M = 5\log(d_L) - 5 \tag{85}$$

because this relation must also hold for the luminosity distance. Because we don't have a nice expression for d_L, we also don't have one for the distance modulus. The book gives an approximate expression that is illustrative though:

$$m - M \simeq 42.38 - 5\log(h) + 5\log(z) + 1.086(1 - q_0)z \tag{86}$$

This is only strictly vald for $z << 1$. q_0 is the deceleration parameter we had before:

$$q(t) = -\frac{R(t)[d^2R(t)/dt^2]}{[dR(t)/dt]^2} \tag{87}$$

This shows that when z is small, the relationship approaches $m - M \propto \log(h)$. As z increases, the final term becomes dominant. This is due to the cosmological constant.

So all we have to do is measure the distance modulus and compare it with the predictions from various Universe models. When we do this we find that SN Type Ia are systematically fainter than we thought, by about 0.25 magnitudes. Thus, they are further away than would be expected for a Universe with $q = 0$, or the Universe has been accelerating in its expansion. The fate of the Universe will be to expand forever at an ever=increasing rate, ending in a Big Freeze.

Distances in Cosmology

Since the Universe itself is expanding, we have many choices of our preferred distance measure. Here, let's discuss the most common ones.

First, we need to introduce the Robertson-Walker metric, which defines a spacetime interval between two events in an isotropic, homogeneous Universe.

$$(ds)^2 = -(c\,dt)^2 + R^2(t)\left[\left(\frac{d\varpi}{1 - k\varpi^2}\right)^2 + (\varpi d\theta)^2 + (\varpi \sin\theta d\phi)^2\right] \tag{88}$$

We won't go into much detail on this equation, but we do need to explain ϖ, which has the same definition as before. This is our "comoving coordinate" and is unchanged with Universe expansion. t is the universal time since the Big Bang.

For light, $ds = 0$ and for a radial path $d\theta = d\phi = 0$ so therefore

$$\frac{c\,dt}{R(t)} = \frac{d\varpi}{\sqrt{1 - k\varpi^2}} \tag{89}$$

Proper Distance

Proper distance roughly corresponds to where a distant object would be at a specific moment of cosmological time, which can change over time due to the expansion of the universe. Importantly, it factors in the expansion.

$$d_P(t) = R(t)\int_0^{\varpi} \frac{d\varpi'}{\sqrt{1 - k\varpi^2}} \tag{90}$$

We can integrate the radial light travel path from observed to emitted time to get

$$\int_{t_e}^{t_0} \frac{c\,dt}{R(t)} = \int_0^{\varpi_e} \frac{d\varpi'}{\sqrt{1 - k\varpi^2}} \tag{91}$$

And combining these find

$$d_P(t) = R(t) \int_{t_e}^{t_0} \frac{c\,dt'}{R(t')} \tag{92}$$

Importantly, $d_{P,0} = d_P(t_0)$ is the distance to ab object *today* and not when the light was emitted.

Today, $d_{P,0} = \varpi$ if $k = 0$ but coordinate and proper distances disagree if $k \neq 0$.

Angular Diameter Distance

Imagine a source of size D that that subtends $\delta\theta$. Then

$$d_A \equiv \frac{D}{\delta\theta} \tag{93}$$

The angular diameter distance obviously depends on the expansion of the Universe. As the universe expands, it increases. Astronomers thought that if they were able to compare the angular diameter distance of an object in the present and past, they could determine how much expansion has taken place. Because there are no such objects that don't also evolve (i.e., galaxies today are different than they were in the past), this method doesn't have much accuracy.

We won't derive it, but we can rewrite the angular diameter distancee as

$$d_A(z) = \frac{c}{H_0} \frac{1}{1+z} \int_0^z \frac{dz}{[\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}]^{0.5}} \tag{94}$$

In this expression, we have ignored the very small $\Omega_{r,0}$.

Luminosity Distance

The luminosity distance is the distance that relates the flux and luminosity:

$$d_L^2 \equiv \frac{L}{4\pi F} \tag{95}$$

The flux decreases as $1/\varpi^2$. Because we are dealing with flux, we also have to account for how light is affacted. Cosmological redshift reduces the flux by a factor of $1 + z$ and cosmological time delay adds an additional factor of $1 + z$). Thus,

$$F = \frac{L}{4\pi\varpi^2(1+z)^2} \tag{96}$$

so

$$d_L = \varpi(1+z)^2 \tag{97}$$

or

$$d_L = (a+z)^2 d_A \tag{98}$$