ISM HW#5

1) Given $I_\nu - I_{\nu,BG} = (S_\nu - I_{\nu,BG})(1-e^{-\tau_\nu})$

and $T^* = \left[\frac{1}{e^{h\nu/kT_{ex}}-1} - \frac{1}{e^{h\nu/kT_{BG}}-1}\right]\frac{h\nu}{k}$

We know $T_B = \frac{c^2}{2k\nu^2} I_\nu$

where $F_\nu = I_\nu \Omega$ for plane-parallel geometry

$S_\nu = B_\nu(T_{ex}) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT_{ex}}-1}$

Assume BG is black-body.

$I_\nu^{off} = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT_{BG}}-1}$

$I_\nu^{on} = I_\nu^{off} e^{-\tau_\nu} + \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT_{ex}}-1}(1-e^{\tau_\nu})$

$\Delta I_\nu = (1-e^{\tau_\nu})(S_\nu - I_{\nu,BG})$

$= (1-e^{-\tau_\nu})\frac{2h\nu^3}{c^2}\left[\frac{1}{e^{h\nu/kT_{ex}}-1} - \frac{1}{e^{h\nu/kT_{BG}}-1}\right]$

$\Delta T = \frac{h\nu}{k}\left[\frac{1}{e^{h\nu/kT_{ex}}-1} - \frac{1}{e^{h\nu/kT_{BG}}-1}\right](1-e^{-\tau_\nu})$

Solves to

$$T_{ex} = \frac{h\nu/k}{\ln\left[1+(h\nu/k)/(\Delta T + e^{h\nu/kT_{BG}}-1)\right]}$$

b) $$\frac{T^*_{12CO}}{T^*_{13CO}} = \frac{\nu_{12CO}}{\nu_{13CO}} \frac{1 - e^{-\tau_{\nu,12CO}}}{1 - e^{-\tau_{\nu,13CO}}}$$

Assume $\tau_{\nu,12CO} \gg 1$, $e^{-\tau_{\nu,12CO}} \to 0$

$$\Rightarrow 1 - e^{-\tau_{\nu,13CO}} = \frac{\nu_{12CO}}{\nu_{13CO}} \frac{T^*_{13CO}}{T^*_{12CO}}$$

$$\Rightarrow \tau_{\nu,13CO} = -\ln\left[1 - \frac{T^*_{13CO}\,\nu_{12CO}}{T^*_{12CO}\,\nu_{13CO}}\right]$$

c) Ugly! Can write out entire expression from notes or use approximations. In either case $N \propto \int n\,ds \propto \int \tau\,ds$

2) We know

$$\ln\left(\frac{N_J}{2J+1}\right) = \ln\left(\frac{N_T}{Z(T_{ex})}\right) - \frac{E_J}{kT_{ex}}$$

$$\text{slope} = -\frac{1}{T_{ex}} \quad \text{if} \quad x = \frac{E_J}{k} = \frac{hBJ(J+1)}{k} \approx 5.5K \cdot J$$
w/ $J = 1, 10$

So, going through points at $J = 1 \to 0$ and $J = 10 \to 9$

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{\ln\left[\frac{7\times10^{15}\ \text{cm}^{-2}}{3}\right] - \ln\left[\frac{1\times10^{9}\ \text{cm}^{-2}}{21}\right]}{5.5K\,(1-10)}$$

$$= -0.36$$

$$\Rightarrow T_{ex} = -\frac{1}{-0.36} \qquad T_{ex} \approx 3K$$

J	N_J	$\ln(N_J/2J+1)$	T_{ex}
$1\to0$	7×10^{15}	35.4	
			4.3
$4\to3$	5×10^{14}	31.6	
			2.7
$7\to6$	2×10^{12}	25.6	
			2.1
$10\to9$	1×10^{9}	17.7	

Seems that T_{ex} decreases with increasing J, but hard to say with only four points

3) If we assume CO is stationary because it is massive,

$C = n \sigma v$

Both molecules will have similar σ

$\sigma_{CO} = \pi \cdot (1.128 Å)^2$

If $v = v_{rms} = \sqrt{\frac{3kT}{m}}$

$m = 2 amu = 3.32 \times 10^{-24} g$

$\Rightarrow v = 3.5 \times 10^{4} cm/s$

$C = n \cdot 1.4 \times 10^{-11} cm^3/s$

b) n_{crit} when $A_{ul} = C_{ul}$

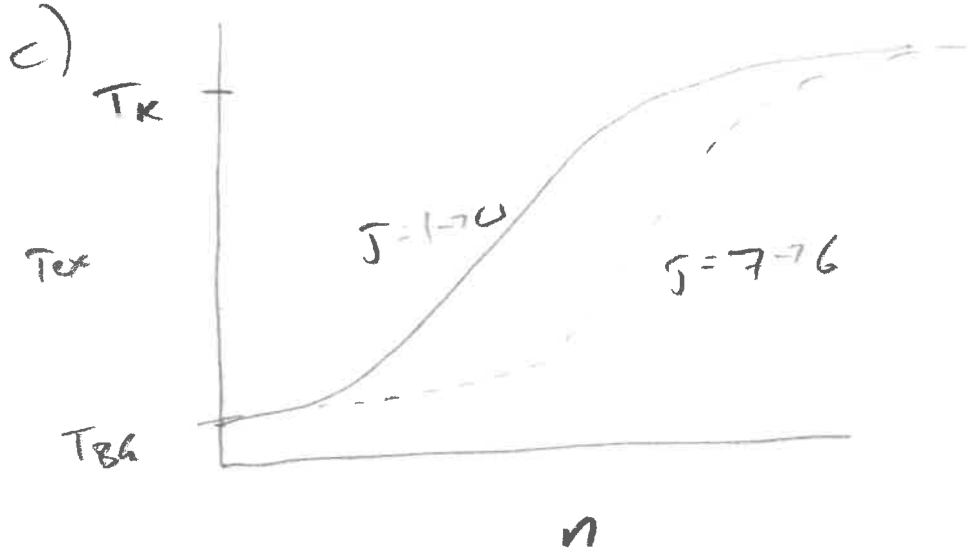
$A_{ul} = n_{crit} \sigma v = n_{crit} \, 1.4 \times 10^{-11} cm^3/s$

$\Rightarrow n_{crit} = \frac{A_{ul}}{1.4 \times 10^{-11} cm^3/s}$

$= \frac{7.26 \times 10^{-8} s^{-1}}{1.4 \times 10^{-11} cm^3/s} = 5.1 \times 10^{3} cm^{-3}$ $J = 1 \to 0$

$= \frac{2.83 \times 10^{-5} s^{-1}}{1.4 \times 10^{-11} cm^3/s} = 2.0 \times 10^{6} cm^{-3}$ $J = 7 \to 6$

c)



d)

$$\frac{n_J}{n_T} = \frac{(2J+1)\, e^{-h\nu/kT_{ex}}}{\sum_{J=0}^{\infty} (2J+1)\, e^{-h\nu/kT_{ex}}}$$

$$\frac{h\nu}{k} \simeq 5.5K \cdot J$$

$$\frac{n_J}{n_T} = \frac{(2J+1)\, e^{-5.5K \cdot J/T_{ex}}}{\sum_{J=0}^{\infty} (2J+1)\, e^{-5.5K \cdot J/T_{ex}}}$$

important to start at J=0!